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NONCOMMUTATIVE BURKHOLDER/ROSENTHAL INEQUALITIES II: APPLICATIONS

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ABSTRACT

We show norm estimates for the sum of independent random variables in noncommutative L_p-spaces for $1 < p < \infty$, following our previous work. These estimates generalize the classical Rosenthal inequality in the commutative case. As applications, we derive an equivalence for the p-norm of the singular values of a random matrix with independent entries, and characterize those symmetric subspaces and unitary ideals which can be realized as subspaces of a noncommutative L_p for $2 < p < \infty$.

0. Introduction and preliminaries

This paper is a continuation of our previous work [JX1] on the investigation of noncommutative martingale inequalities. The classical theory of martingale

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inequalities has a long tradition in probability. It is well-known today that the applications of the works of Burkholder and his collaborators range from classical harmonic analysis to stochastic differential equations and the geometry of Banach spaces. When proving the estimates for the conditioned (or little) square function (cf. [Bu, BuG]), Burkholder was aware of Rosenthal's result [Ro] on sums of independent random variables. Here we proceed differently and prove the noncommutative Rosenthal inequality along the same line as the noncommutative Burkholder inequality from [JX1]. This slightly modified proof yields a better constant. The main intention of this paper is to illustrate the usefulness of the conditioned square function by several examples. For many applications it is important to consider generalized notions of independence. This will allow us to explore applications toward random matrices and symmetric subspaces of noncommutative L_p-spaces.

Our estimates on random matrices are motivated by the following noncommutative Khintchine inequality of Lust-Piquard [LP]. Let (ε_{ij}) be an independent Rademacher family on a probability space (Ω,μ) and let (e_{ij}) be the canonical matrix units of $B(\ell_2)$. Then for any $2\le p<\infty$ there exists a positive constant c_p, depending only on p, such that for scalar coefficients (a_{ij})

$$\mathbb{E}\Big\|\sum_{ij}\varepsilon_{ij}\,a_{ij}\,e_{ij}\Big\|_{S_p} \sim_{c_p} \max\Big\{\Big(\sum_i\Big(\sum_j|a_{ij}|^2\Big)^{p/2}\Big)^{1/p};\Big(\sum_j\Big(\sum_i|a_{ij}|^2\Big)^{p/2}\Big)^{1/p}\Big\},$$

where S_p denotes the usual Schatten p-class. Recall that for a matrix $a=(a_{ij})$

$$\|a\|_{S_p}=\Big[\sum_n\lambda_n(|a|)^p\Big]^{1/p},$$

where the $\lambda_n(|a|)$ are the eigenvalues of $|a|$, arranged in decreasing order and counted according to their multiplicities. In the noncommutative setting it is natural to replace (ε_{ij}) by a noncommutative independent family and the scalar coefficients a_{ij} by operator coefficients. Here we just mention, for illustration, the following special case and refer to section 3 for more information. Let $(f_{ij})\subset L_p(\Omega,\mu)$ be a matrix of independent mean zero random variables. Then for $2 \le p < \infty$

$$\Big\| \sum_{ij} f_{ij} \otimes e_{ij} \Big\|_{L_p(\Omega;S_p)} \sim_{c\,p}$$

$$\max \Big\{ \Big(\sum_{ij} \|f_{ij}\|_p^p \Big)^{1/p}, \Big(\sum_i \Big(\sum_j \|f_{ij}\|_2^2 \Big)^{p/2} \Big)^{1/p}, \Big(\sum_j \Big(\sum_i \|f_{ij}\|_2^2 \Big)^{p/2} \Big)^{1/p} \Big\}$$

and for $p < 2$ (with p' denoting the conjugate index of p)

$$\Big\| \sum_{ij} f_{ij} \otimes e_{ij} \Big\|_{L_p(\Omega;S_p)} \sim_{c\,p'}$$

$$\inf \Big\{ \Big(\sum_{ij} \|d_{ij}\|_p^p \Big)^{1/p} + \Big(\sum_i \Big(\sum_j \|g_{ij}\|_2^2 \Big)^{p/2} \Big)^{1/p} + \Big(\sum_j \Big(\sum_i \|h_{ij}\|_2^2 \Big)^{p/2} \Big)^{1/p} \Big\},$$

where the infimum is taken over all decompositions $f_{ij} = d_{ij} + g_{ij} + h_{ij}$ with mean zero variables d_{ij}, g_{ij} and h_{ij}, which, for each couple (i, j), are measurable with respect to the σ-algebra generated by f_{ij}.

The equivalence above, for $p \ge 2$, is a direct consequence of our noncommutative Rosenthal inequality in Section 2. As usual, the case $p < 2$ is dealt with by duality. Sections 2 and 3 are devoted to the Rosenthal inequalities for $p \ge 2$ and $p < 2$, respectively. The random variables we consider are general independent variables in noncommutative L_p-spaces (including the type III case). In contrast with the classical case where there exist a unique independence, one has several different notions of independence in the noncommutative setting. Introduced in Section 1, our definition of independence embraces the most commonly used noncommutative notions of independence. These include the usual tensor independence and Voiculescu's freeness.

In the light of the recent concept of noncommutative maximal functions, it would be desirable to have a perfect noncommutative analogue of the classical Burkholder inequality by replacing the diagonal term $\|(d_k)\|_{\ell_p(L_p)}$ by the maximal term $\|(d_k)\|_{L_p(\ell_\infty)}$. This is indeed possible. We will make up for it in Section 4. The same variant is, of course, true for the noncommutative Rosenthal inequality.

Symmetric subspaces of L_p-spaces are motivated by probabilistic notions of exchangeable random variables. In the commutative situation, the memoir of Johnson, Maurey, Schechtman and Tzafriri [JMST] contains an impressive amount of information and many sophisticated applications of probabilistic techniques. As applications of the noncommutative Burkholder/Rosenthal inequalities, we will extend some of their results to the noncommutative setting in section 6. Below is an elementary example. Let $\mathcal{A}$ and $\mathcal{M}$ be von Neumann algebras and $2 \le p < \infty$. Let $(x_k)_{1 \le k \le n} \subset L_p(\mathcal{M})$ and $\lambda > 0$ such that

$$\Big\| \sum_{k=1}^n \varepsilon_k a_{\pi(k)} \otimes x_k \Big\|_p \le \lambda \Big\| \sum_{k=1}^n a_k \otimes x_k \Big\|_p$$

holds for all $\varepsilon_k = \pm 1$, all permutations π on $\{1, \dots, n\}$ and coefficients $a_k \in L_p(\mathcal{A})$. Then there are constants α, β and γ, depending only on (x_k), such that for all $a_k \in L_p(\mathcal{A})$

$$\Big\| \sum_{k=1}^n a_k \otimes e_k \Big\|_p \sim_{c_{p,\lambda}} \max \Big\{ \alpha \Big(\sum_{k=1}^n \|a_k\|_p^p \Big)^{1/p}, \beta \Big\| \Big(\sum_{k=1}^n a_k^* a_k \Big)^{1/2} \Big\|_p, \gamma \Big\| \Big(\sum_{k=1}^n a_k a_k^* \Big)^{1/2} \Big\|_p \Big\}.$$

As a consequence of this statement (with $\mathcal{A} = \mathbb{C}$), we deduce that ℓ_p and ℓ_2 are the only Banach spaces with a symmetric basis embedding into a noncommutative L_p for $2 < p < \infty$. On the other hand, at the operator space level, we have four spaces ℓ_p, C_p, R_p and $C_p \cap R_p$, where C_p and R_p are respectively the column and row subspaces of S_p. In the same spirit, we also characterize the unitary ideals isomorphic to subspaces of a noncommutative L_p for $2 < p < \infty$ in Section 7.

In the remainder of this introduction we give some necessary preliminaries and notation. We use standard notation from von Neumann algebra theory (see, e.g., [KR, T2, St]). For noncommutative L_p-spaces we follow the notation system of [JX1], and refer there for more details and all unexplained notions, especially those on martingales. As in [JX1], the noncommutative L_p-spaces used in this paper are those constructed by Haagerup [H1]. We will work under the standard assumptions from [JX1]. In particular, $\mathcal{M}$ is a σ-finite von Neumann algebra equipped with a normal faithful state φ. The Haagerup noncommutative L_p-spaces associated with $(\mathcal{M}, \varphi)$ are denoted by $L_p(\mathcal{M})$. We denote by D the density of φ in the space $L_1(\mathcal{M})$ such that

$$\varphi(x) = \mathrm{tr}(xD), \quad x \in \mathcal{M},$$

where $\mathrm{tr} : L_1(\mathcal{M}) \to \mathbb{C}$ is the distinguished tracial functional. The norm of $L_p(\mathcal{M})$ is denoted by $\| \, \|_p$. Recall that $\mathcal{M}D^{1/p}$ is dense in $L_p(\mathcal{M})$ for any $0 < p < \infty$. More generally, $D^{(1-\theta)/p}\mathcal{M}_a D^{\theta/p}$ is also dense in $L_p(\mathcal{M})$ for any $0 \leq \theta \leq 1$, where $\mathcal{M}_a$ denotes the family of all analytic elements with respect to the modular group σ_t^φ of φ.

Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$ (i.e., a w*-closed involutive subalgebra containing the unit of $\mathcal{M}$). We say that $\mathcal{N}$ is **φ-invariant** if $\sigma_t^\varphi(\mathcal{N}) \subset \mathcal{N}$ for all $t \in \mathbb{R}$. According to Takesaki [T1], there exists a unique normal faithful conditional expectation $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ such that $\varphi \circ \mathcal{E} = \varphi$. Recall that $\mathcal{E}$ is characterized by

$$\varphi(\mathcal{E}(x)y) = \varphi(xy)\,, \quad x \in \mathcal{M},\ y \in \mathcal{N}.$$

Note that $\mathcal{E}$ commutes with the modular group σ_t^φ of φ. Namely, $\sigma_t^\varphi \circ \mathcal{E} = \mathcal{E} \circ \sigma_t^\varphi$. In these circumstances, $\sigma_t^\varphi\big|_{\mathcal{N}}$ is the modular group of $\varphi\big|_{\mathcal{N}}$, and the noncommutative $L_p(\mathcal{N})$ associated to $(\mathcal{N}, \varphi\big|_{\mathcal{N}})$ can be naturally isometrically identified with a subspace of $L_p(\mathcal{M})$. With this identification, the density of $\varphi\big|_{\mathcal{N}}$ in $L_1(\mathcal{N})$ coincides with D. All these allow us to not distinguish φ, σ_t^φ and D and their respective restrictions to $\mathcal{N}$.

For $1 \leq p < \infty$, the conditional expectation $\mathcal{E}$ extends to a contractive projection $\mathcal{E}_p$ from $L_p(\mathcal{M})$ onto $L_p(\mathcal{N})$ densely defined by

$$\mathcal{E}_p(xD^{1/p}) = \mathcal{E}(x)D^{1/p}\,, \quad x \in \mathcal{M}.$$

$\mathcal{E}_p$ is also determined by

$$\mathcal{E}_p(D^{(1-\theta)/p}xD^{\theta/p}) = D^{(1-\theta)/p}\mathcal{E}(x)D^{\theta/p}\,, \quad x \in \mathcal{M}_a\,,\ 0 \leq \theta \leq 1\,.$$

It is convenient to drop the index p. This is also justified by using Kosaki's embedding $I : L_p(\mathcal{M}) \to L_1(\mathcal{M})$, $I(xD^{1/p}) = xD$ since then $\mathcal{E}_1(I(y)) = I(\mathcal{E}_p(y))$. In this sense all maps $\mathcal{E}_p$ are induced by the same map $\mathcal{E}_1$.

Recall that if $\mathcal{N} = \mathbb{C}$, then $\mathcal{E}(x) = \varphi(x)1$ for every $x \in \mathcal{M}$; so $\mathcal{E}$ can be identified with φ. The action of $\mathcal{E}$ on $L_p(\mathcal{M})$ is then given by $\mathcal{E}(x) = \mathrm{tr}(xD^{1/p'})D^{1/p}$, where p' denotes the conjugate index of p. Thus if additionally φ is tracial, we still have $\mathcal{E}(x) = \varphi(x)1$ for $x \in L_p(\mathcal{M})$.

We will frequently use the column, row spaces and their conditional versions. Recall that for a finite sequence $a = (a_k) \subset L_p(\mathcal{M})$

$$\|a\|_{L^p(\mathcal{M};\ell_2^c)} = \Big\| \Big(\sum_k |a_k|^2 \Big)^{1/2} \Big\|_p \quad \text{and} \quad \|a\|_{L_p(\mathcal{M};\ell_2^r)} = \Big\| \Big(\sum_k |a_k^*|^2 \Big)^{1/2} \Big\|_p .$$

Then $L_p(\mathcal{M};\ell_2^c)$ and $L_p(\mathcal{M};\ell_2^r)$ are the completions of the family of all finite sequences in $L_p(\mathcal{M})$ with respect to $\| \ \|_{L_p(\mathcal{M};\ell_2^c)}$ and $\| \ \|_{L_p(\mathcal{M};\ell_2^r)}$, respectively (in the w*-topology for $p = \infty$). It is convenient to view $L_p(\mathcal{M};\ell_2^c)$ and $L_p(\mathcal{M};\ell_2^r)$ as the first column and row subspaces of $L_p(B(\ell_2)\bar{\otimes}\mathcal{M})$, respectively.

Now let $\mathcal{N}$ be a φ-invariant von Neumann subalgebra of $\mathcal{M}$ with conditional expectation $\mathcal{E}$. Let $p \geq 2$ and $a = (a_k) \subset L_p(\mathcal{M})$ be a finite sequence. Since $a_k^* a_k \in L_{p/2}(\mathcal{M})$ and $p/2 \geq 1$, $\mathcal{E}(a_k^* a_k)$ is well-defined and we can consider

$$\|a\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^c)} = \Big\| \Big(\sum_k \mathcal{E}(a_k^* a_k) \Big)^{1/2} \Big\|_p .$$

According to [J1] (see also [JX1]), this defines a norm on the family of all finite sequences in $L_p(\mathcal{M})$. The corresponding completion (relative to the w*-topology for $p = \infty$) is the conditional column space $L_p(\mathcal{M},\mathcal{E};\ell_2^c)$. Note that if $2 \leq p < \infty$, then finite sequences in $\mathcal{M}_a D^{1/p}$ are dense in $L_p(\mathcal{M},\mathcal{E};\ell_2^c)$. The latter density allows us to extend the definition to the range $1 \leq p < 2$. Let $a = (a_k) \subset \mathcal{M} D^{1/p}$ with $a_k = b_k D^{1/p}$, $b_k \in \mathcal{M}$. Set

$$\|a\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^c)} = \Big\| \Big(\sum_k D^{1/p}\, \mathcal{E}(b_k^* b_k) D^{1/p} \Big)^{1/2} \Big\|_p .$$

We have again a norm. The resulting completion is denoted by $L_p(\mathcal{M},\mathcal{E};\ell_2^c)$. The conditional row space $L_p(\mathcal{M},\mathcal{E};\ell_2^r)$ is defined as the space of all (a_k) such that $(a_k^*) \in L_p(\mathcal{M},\mathcal{E};\ell_2^c)$, equipped with the norm

$$\|(a_k)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^r)} = \|(a_k^*)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^c)} .$$

The space $L_p(\mathcal{M},\mathcal{E};\ell_2^c)$ (resp., $L_p(\mathcal{M},\mathcal{E};\ell_2^r)$) can be equally viewed as the first column (resp., row) subspace of $L_p(B(\ell_2(\mathbb{N}^2))\bar{\otimes}\mathcal{M})$, indexed by a double index.

Lemma 0.1: *Let $1 \leq p < \infty$ and p' be the index conjugate to p. Then*

$$L_p(\mathcal{M},\mathcal{E};\ell_2^c)^* = L_{p'}(\mathcal{M},\mathcal{E};\ell_2^c)$$

holds isometrically with respect to the antilinear duality bracket:

$$\langle a, b \rangle = \sum \mathrm{tr}(b_k^* a_k)\,, \quad a \in L_p(\mathcal{M},\mathcal{E};\ell_2^c),\ b \in L_{p'}(\mathcal{M},\mathcal{E};\ell_2^c).$$

A similar statement holds for the conditional row spaces.

Proof. This is the column (or row) space version of [J1, Corollary 2.12]. The proof there can be adapted to the present situation by considering $\mathcal{M}\bar{\otimes}B(\ell_2)$ and $\mathcal{N}\bar{\otimes}B(\ell_2)$ in place of $\mathcal{M}$ and $\mathcal{N}$, respectively. It then remains to note that the column space $L_p(\mathcal{M};\ell_2^c)$ is complemented in $L_p(B(\ell_2)\bar{\otimes}\mathcal{M})$. See also the proof of [J1, Theorem 2.13], where instead of one conditional expectation, a sequence of conditional expectations is involved (then the noncommutative Stein inequality is needed). We omit the details. ∎

The preceding notation will be kept in the remainder of the paper. Unless explicitly stated otherwise, $\mathcal{M}$ will denote a von Neumann algebra equipped with a normal faithful state φ. If $\mathcal{N}$ is a φ-invariant von Neumann subalgebra of $\mathcal{M}$, its associated conditional expectation will be often denoted by $\mathcal{E}_{\mathcal{N}}$ or simply by $\mathcal{E}$ if no confusion can occur.

The first version of this paper was written immediately after the submission of [JX1] (more than five years ago). Since then considerable progress has been made on noncommutative martingale inequalities. We mention only [JX2, PaR, R2, R3, R4], where, among many other results, the optimal orders of the best constants in most noncommutative martingale inequalities are determined.

1. Independence

In this section, we first introduce the central notion for our formulation of the noncommutative Rosenthal inequality, i.e., the independence. We then present some natural examples of noncommutative independent variables. Our setup is the following: $\mathcal{N}$ and $\mathcal{A}_k$ are φ-invariant von Neumann subalgebras of $\mathcal{M}$ such that $\mathcal{N}\subset\mathcal{A}_k$ for every k. The sequence $(\mathcal{A}_k)$ can be finite.

(I) We say that $(\mathcal{A}_k)$ are (**faithfully**) **independent over** $\mathcal{N}$ or **with respect to** $\mathcal{E}_{\mathcal{N}}$ if for every k, $\mathcal{E}_{\mathcal{N}}(xy)=\mathcal{E}_{\mathcal{N}}(x)\mathcal{E}_{\mathcal{N}}(y)$ holds for all $x\in\mathcal{A}_k$ and y in the von Neumann subalgebra generated by $(\mathcal{A}_j)_{j\neq k}$.

(II) We say that $(\mathcal{A}_k)$ are (**faithfully**) **order independent** over $\mathcal{N}$ or **with respect to** $\mathcal{E}_{\mathcal{N}}$ if for every $k\geq 2$, $\mathcal{E}_{VN(\mathcal{A}_1,\ldots,\mathcal{A}_{k-1})}(x)=\mathcal{E}_{\mathcal{N}}(x)$ holds for all $x\in\mathcal{A}_k$, where $VN(\mathcal{A}_1,\ldots,\mathcal{A}_{k-1})$ denotes the von Neumann subalgebra generated by $\mathcal{A}_1,\ldots,\mathcal{A}_{k-1}$.

(III) A sequence $(x_k) \subset L_p(\mathcal{M})$ is said to be **faithfully (order) independent with respect to** $\mathcal{E}_{\mathcal{N}}$ if there exist $\mathcal{A}_k$ such that $x_k \in L_p(\mathcal{A}_k)$ and $(\mathcal{A}_k)$ is faithfully (order) independent with respect to $\mathcal{E}_{\mathcal{N}}$.

Note that the subalgebra $VN(\mathcal{A}_1, \dots, \mathcal{A}_{k-1})$ is φ-invariant too, so the conditional expectation $\mathcal{E}_{VN(\mathcal{A}_1,\dots,\mathcal{A}_{k-1})}$ exists. Also note that the independence in (I) can be defined for any family (without order). The adverb **faithfully** refers to the faithfulness of the state φ. We will also consider the nonfaithful case in Section 5. If no confusion can occur, we will often drop this adverb by saying simply independent or order independent. If $\mathcal{N} = \mathbb{C}$, these notions are, of course, with respect to the state φ

Remark 1.1: Let $(\mathcal{A}_k)$ be order independent over $\mathcal{N}$. Then for every k

$$\mathcal{E}_{VN(\mathcal{A}_1,\dots,\mathcal{A}_{k-1})}(x) = \mathcal{E}_N(x), \quad x \in \mathcal{A}_j,\ j \geq k.$$

Indeed, we have

$$\begin{aligned}\mathcal{E}_{VN(\mathcal{A}_1,\dots,\mathcal{A}_{k-1})}(x) &= \mathcal{E}_{VN(\mathcal{A}_1,\dots,\mathcal{A}_{k-1})}\big(\mathcal{E}_{VN(\mathcal{A}_1,\dots,\mathcal{A}_{j-1})}(x)\big)\\ &= \mathcal{E}_{VN(\mathcal{A}_1,\dots,\mathcal{A}_{k-1})}\big(\mathcal{E}_{\mathcal{N}}(x)\big) = \mathcal{E}_{\mathcal{N}}(x).\end{aligned}$$

It follows that if $x_k \in L_p(\mathcal{A}_k)$ with $\mathcal{E}_{\mathcal{N}}(x_k) = 0$, then (x_k) is a martingale difference sequence with respect to the filtration $\big(VN(\mathcal{A}_1, \dots, \mathcal{A}_k)\big)_{k\geq 1}$.

Lemma 1.2: *Assume that* $(\mathcal{A}_k)$ *is independent over* $\mathcal{N}$.

(i) $(\mathcal{A}_k)$ *is order independent over* $\mathcal{N}$.

(ii) *If* $x_k \in L_p(\mathcal{A}_k)$ *satisfy* $\mathcal{E}_N(x_k) = 0$, $1 \leq p \leq \infty$, *then*

$$\Big\| \sum_{k=1}^n \varepsilon_k x_k \Big\|_p \leq 2 \Big\| \sum_{k=1}^n x_k \Big\|_p, \quad \varepsilon_k = \pm 1.$$

Proof. Let S be a subset of indices and $\mathcal{B}_S = VN\{\mathcal{A}_j : j \in S\}$. Set $\mathcal{E}_S = \mathcal{E}_{\mathcal{B}_S}$. Fix $k \notin S$. Now let $x \in \mathcal{A}_k$. We want to prove $\mathcal{E}_S(x) = \mathcal{E}_{\mathcal{N}}(x)$. For this, it suffices to show

$$\varphi(xy) = \varphi(\mathcal{E}_{\mathcal{N}}(x)y), \quad y \in \mathcal{B}_S\,.$$

This equality immediately follows from the independence of $(\mathcal{A}_k)$ over $\mathcal{N}$ for

$$\varphi(xy) = \varphi(\mathcal{E}_{\mathcal{N}}(xy)) = \varphi(\mathcal{E}_{\mathcal{N}}(x)\mathcal{E}_{\mathcal{N}}(y)) = \varphi(\mathcal{E}_{\mathcal{N}}(\mathcal{E}_{\mathcal{N}}(x)y)) = \varphi(\mathcal{E}_{\mathcal{N}}(x)y).$$

If we apply this to the subset $S = \{1, \dots, k-1\}$, we obtain (i). To prove the second assertion, consider $\varepsilon_k = \pm 1$ and define $S = \{k : \varepsilon_k = 1\}$. By approximation by elements of the form $x_k = a_k D^{1/p}$, $a_k \in \mathcal{A}_k$ and $\mathcal{E}_N(a_k) = 0$, we see that

$$\mathcal{E}_S\Big(\sum_{k=1}^n x_k\Big) = \sum_{k\in S} x_k + \sum_{k\notin S} \mathcal{E}_S(x_k) = \sum_{k\in S} x_k.$$

Since $\mathcal{E}_S$ is a contraction on $L_p(\mathcal{M})$,

$$\Big\|\sum_{k\in S} x_k\Big\|_p = \Big\|\mathcal{E}_S\Big(\sum_{k=1}^n x_k\Big)\Big\|_p \le \Big\|\sum_{k=1}^n x_k\Big\|_p;$$

whence

$$\Big\|\sum_{k=1}^n \varepsilon_k x_k\Big\|_p \le \Big\|\sum_{k\in S} x_k\Big\|_p + \Big\|\sum_{k\in S^c} x_k\Big\|_p \le 2\|\sum_{k=1}^n x_k\|_p \,. \qquad \blacksquare$$

In the rest of this section we give some natural examples of independent variables, which often occur in noncommutative probability.

Example 1.3: Classical independence. Let (Ω, μ) be a probability space, and let $(\mathcal{N}, \psi)$ be a von Neumann algebra equipped with a normal faithful state ψ. Let $\mathcal{M} = L_\infty(\Omega)\bar{\otimes}\mathcal{N}$ be the von Neumann algebra tensor product equipped with the tensor product state $\varphi = \mu \otimes \psi$. We view $\mathcal{N}$ as a subalgebra of $\mathcal{M}$ in the natural way. Then the conditional expectation $\mathcal{E}_\mathcal{N}$ is given by

$$\mathcal{E}_\mathcal{N}(x) = \int_\Omega x d\mu\,, \quad x \in \mathcal{M}\,,$$

where the integral is taken with respect to the w*-topology of $\mathcal{M}$. Also recall that the noncommutative L_p-space $L_p(\mathcal{M})$ coincides with the usual L_p-space $L_p(\Omega; L_p(\mathcal{N}))$ of p-integrable functions on Ω with values in $L_p(\mathcal{N})$. In this case, the independence with respect to $\mathcal{E}_\mathcal{N}$ coincides with the classical independence of vector-valued random variables. In particular, if $(f_n) \subset L_p(\Omega)$ is an independent sequence of random variables in the usual sense, then $(f_n a_n)$ is independent with respect to $\mathcal{E}_\mathcal{N}$ for any $(a_n) \subset L_p(\mathcal{N})$.

Example 1.4: Tensor independence. This independence is the most transparent generalization of the classical one to the noncommutative setting. Let $(\mathcal{A}_k, \varphi_k)$ be a sequence of von Neumann algebras equipped with normal faithful states φ_k. Let

$$(\mathcal{M}, \varphi) = \overline{\bigotimes_{k\ge 0}} (\mathcal{A}_k, \varphi_k)$$

denote the corresponding von Neumann algebra tensor product. As usual, we regard $\mathcal{A}_k$ as von Neumann subalgebras of $\mathcal{M}$. It is clear that they are φ-invariant. The conditional expectation $\mathcal{E}_{\mathcal{A}_k}$ is uniquely determined by

$$\mathcal{E}_{\mathcal{A}_k}(a_0 \otimes \cdots \otimes a_m) = \Big[\prod_{j\neq k} \varphi_j(a_j)\Big]\, a_k, \quad m \geq 0.$$

Clearly, $(\mathcal{A}_k)_{k\geq 1}$ is independent over $\mathcal{A}_0$. If all $\mathcal{A}_k$ are commutative, we go back to the classical case.

Example 1.5: Free independence. Our reference for this example is [VDN]. Let $(\mathcal{A}_k)_{k\geq 1}$ be a sequence of von Neumann subalgebras of $\mathcal{M}$, and let $\mathcal{B}$ be a common von Neumann subalgebra of the $\mathcal{A}_k$. Assume that there exist normal faithful conditional expectations $\mathcal{E} : \mathcal{M} \to \mathcal{B}$ and $\mathcal{E}_k : \mathcal{A}_k \to \mathcal{B}$. The sequence $(\mathcal{A}_k)_{k\geq 1}$ is called free over $\mathcal{B}$ if

$$\mathcal{E}(x_1 \cdots x_k) = 0$$

whenever $x_j \in \mathring{\mathcal{A}}_{i_j}$ and $i_1 \neq i_2 \neq \cdots \neq i_k$, where $\mathring{\mathcal{A}}_k = \ker \mathcal{E}_k$. If $\mathcal{B} = \mathbb{C}$, we get the freeness with respect to the state $\varphi \sim \mathcal{E}$. There exists an equivalent way of formulating freeness by using reduced free product. Without loss of generality we may assume that $\mathcal{M}$ is generated by the $\mathcal{A}_k$. Then $(\mathcal{M}, \mathcal{E})$ can be identified with the von Neumann algebra amalgamated reduced free product of the $(\mathcal{A}_k, \mathcal{E}_k)$:

$$(\mathcal{M}, \mathcal{E}) = \mathop{\bar{*}_{\mathcal{B}}}_{k\geq 1} (\mathcal{A}_k, \mathcal{E}_k).$$

Assume in addition that $\mathcal{B}$ is σ-finite, and fix a normal faithful state ϕ on $\mathcal{B}$. Then $\varphi = \phi \circ \mathcal{E}$ is a normal faithful state on $\mathcal{M}$ and the $\mathcal{A}_k$ are φ-invariant. One easily checks that freeness implies the independence in our sense.

Let us consider the particularly interesting case where all $\mathcal{A}_k$ are equal to $L_\infty(-2, 2)$, equipped with the Wigner measure

$$d\mu(t) = \frac{1}{2\pi}\sqrt{4 - t^2}\, dt.$$

Then the reduced free product (without amalgamation)

$$(\mathcal{M}, \varphi) = \mathop{\bar{*}}_{k\geq 1} \mathcal{A}_k$$

is a II_1 factor with φ a normal faithful tracial state. Let $x_k \in \mathcal{A}_k$ be given by $x_k(t) = t$. Then the sequence (x_k) is free. This is a semicircular system in Voiculescu's sense. It is the free analogue of a standard Gaussian system.

Semicircular systems admit a more convenient realization via Fock spaces. Let us describe this briefly. Let H be a complex Hilbert space. The associated free (or full) Fock space is defined by

$$\mathcal{F}(H) = \bigoplus_{n\geq 0} H^{\otimes n},$$

where $H^{\otimes 0} = \mathbb{C}\mathbb{1}$ ($\mathbb{1}$ being a unit vector, called vacuum), and $H^{\otimes n}$ is the n-th Hilbertian tensor power of H for $n \geq 1$. The (left) creator associated with a vector $\xi \in H$ is the operator on $\mathcal{F}(H)$ uniquely determined by

$$c(\xi)\, \xi_1 \otimes \cdots \otimes \xi_n = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n$$

for any $\xi_1, \ldots, \xi_n \in H$. Here $\xi_1 \otimes \cdots \otimes \xi_n$ is understood as the vacuum $\mathbb{1}$ if $n = 0$. Its adjoint is given by

$$c(\xi)^*\, \xi_1 \otimes \cdots \otimes \xi_n = \langle \xi_1,\ \xi\rangle\, \xi_2 \otimes \cdots \otimes \xi_n$$

(with $c(\xi)^*\mathbb{1} = 0$). This is the annihilator associated with ξ and is denoted by $a(\xi)$. We have the following free commutation relation:

$$a(\eta)c(\xi) = \langle \xi,\ \eta\rangle 1, \quad \xi,\ \eta \in H.$$

Now assume that H is the complexification of a real Hilbert space $H_{\mathbb{R}}$. For a real $\xi \in H_{\mathbb{R}}$ define

$$g(\xi) = c(\xi) + a(\xi).$$

Let $\Gamma(H)$ be the von Neumann subalgebra of $B(\mathcal{F}(H))$ generated by all $g(\xi)$ with real $\xi \in H_{\mathbb{R}}$:

$$\Gamma(H) = \big\{g(\xi) : \xi \in H_{\mathbb{R}}\big\}''.$$

This is the free von Neumann algebra associated with H (or more precisely, with $H_{\mathbb{R}}$). The vector state φ defined by the vacuum, $x \mapsto \langle x\mathbb{1},\ \mathbb{1}\rangle$ is faithful and tracial on $\Gamma(H)$. If (ξ_k) is an orthonormal system of H consisting of real vectors, then $(g(\xi_k))$ is a semicircular system.

The preceding Fock space construction can be deformed to get type III algebras. For this, let H be separable and fix an orthonormal basis $(e_{\pm k})_{k\geq 1}$ of H consisting of real vectors. Let $\lambda = (\lambda_k)$ be a sequence of positive numbers. Set

$$g_k = c(e_k) + \sqrt{\lambda_k}\, a(e_{-k}), \quad k \geq 1. \tag{1.1}$$

Let Γ_λ be the von Neumann algebra on $\mathcal{F}(H)$ generated by (g_k), and let φ_λ be the vector state on Γ_λ determined by the vacuum. Then (g_k) is free in $(\Gamma_\lambda, \varphi_\lambda)$. This is a generalized circular system in Shlyakhtenko's sense [S]. If all λ_k are equal to 1, Γ_λ becomes the previous free von Neumann algebra $\Gamma(H)$ associated with H. Otherwise, Γ_λ is a type III factor and the state φ_λ is called a free quasi-free state.

Example 1.6: q-independence. The Fock space construction in the previous example can be modified to embrace the so-called q-independence, $-1 \leq q \leq 1$, introduced by Bożejko and Speicher [BS1, BS2, BKS]. Again, let H be the complexification of a real Hilbert space $H_{\mathbb{R}}$. The associated q-Fock space $\mathcal{F}_q(H)$ is defined by

$$\mathcal{F}_q(H) = \bigoplus_{n\geq 0} H^{\otimes n},$$

where $H^{\otimes n}$ is now equipped with the q-scalar product for every $n \geq 2$. Recall that $\mathcal{F}_0(H)$ is the free Fock space discussed in the previous example, while $\mathcal{F}_1(H)$ and $\mathcal{F}_{-1}(H)$ are the classical symmetric and antisymmetric Fock spaces, respectively.

Given $\xi \in H$ we define the corresponding creator $c_q(\xi)$ and annihilator $a_q(\xi)$ similarly as in the free case. These are linear operators on $\mathcal{F}_q(H)$ determined by the following conditions

$$c_q(\xi)\, \xi_1 \otimes \cdots \otimes \xi_n = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n$$

and

$$a(\xi)\, \xi_1 \otimes \cdots \otimes \xi_n = \sum_{k=1}^{n} q^{k-1} \langle \xi_k,\, \xi\rangle\, \xi_1 \otimes \cdots \otimes \overset{\vee}{\xi}_k \otimes \cdots \otimes \xi_n,$$

where $\overset{\vee}{\xi}_k$ means that ξ_k is removed from the tensor product. $c_q(\xi)$ and $a_q(\xi)$ are bounded operators if $q < 1$ and closable densely defined operators if $q = 1$. In the latter case, $c_q(\xi)$ and $a_q(\xi)$ also denote their closures. Again, we have $c_q(\xi)^* = a_q(\xi)$. The creators and annihilators satisfy the following q-commutation relations :

$$a_q(\xi)c_q(\eta) - q\, c_q(\eta)a_q(\xi) = \langle \eta,\, \xi\rangle 1, \quad \xi,\, \eta \in H.$$

In the cases of $q = \pm 1$ these are respectively the canonical commutation relations (CCR) and the canonical anticommutation relations (CAR).

Given a real vector $\xi \in H_{\mathbb{R}}$ define

$$g_q(\xi) = c_q(\xi) + a_q(\xi).$$

$g_q(\xi)$ is called a q-Gaussian variable. The q-von Neumann algebra $\Gamma_q(H)$ associated with H is the von Neumann algebra on $\mathcal{F}_q(H)$ generated by the $g_q(\xi)$

with real ξ. As in the free case, the vacuum expectation $x \mapsto \langle x\mathbb{1}, \mathbb{1}\rangle$ is a normal faithful tracial state on $\Gamma_q(H)$, denoted by τ_q. In particular, $\Gamma_0(H)$ is the free von Neumann algebra considered previously. On the other hand, if ξ and η are orthogonal, then $g_1(\xi)$ and $g_1(\eta)$ commute, while $g_{-1}(\xi)$ and $g_{-1}(\eta)$ anticommute. Therefore, $\Gamma_1(H)$ is commutative, while $\Gamma_{-1}(H)$ is a Clifford algebra.

Let $K \subset H$ be a closed subspace, which is the complexification of $K_{\mathbb{R}} \subset H_{\mathbb{R}}$. Then $\Gamma_q(K)$ is a subalgebra of $\Gamma_q(H)$. The associated conditional expectation is given by the second quantization of the orthogonal projection from $H_{\mathbb{R}}$ onto $K_{\mathbb{R}}$. Now let (H_k) be a sequence of subspaces of H which are complexifications of pairwise orthogonal subspaces of $H_{\mathbb{R}}$. Each $\Gamma_q(H_k)$ is identified with the von Neumann subalgebra of $\Gamma_q(H)$ generated by $g_q(\xi)$ with real $\xi \in H_k$. Then the $\Gamma_q(H_k)$ are independent with respect to τ_q. Consequently, if $(\xi_k)_k$ is an orthonormal sequence of real vectors of H, $(g_q(\xi_k))_k$ is independent. This sequence $(g_q(\xi_k))_k$ is called a q-semicircular system.

Shlyakhtenko's generalized circular systems admit q-counterparts too. We refer to [Hi] for more details. Here we briefly discuss only the case $q = -1$, which is a reformulation of the classical construction of the Araki-Woods factors. These latter factors are built using Pauli matrices as follows. We consider the generators of the CAR algebra

$$a_k = 1 \otimes \cdots \otimes 1 \otimes \underbrace{e_{12}}_{k\text{-th position}} \otimes 1 \otimes \cdots \otimes 1 \tag{1.2}$$

in the algebraic tensor product $\bigotimes_{k\geq 1} \mathbb{M}_2$, where, as usual, e_{ij} denote the matrix units of $\mathbb{M}_2 = B(\ell_2^2)$. Fix a sequence $(\mu_k) \subset (0,1)$, and consider the states $\varphi_k = (1-\mu_k)e_{11} + \mu_k e_{22}$ on $\mathbb{M}_2$. Then the tensor product state $\varphi = \bigotimes_{k\geq 1} \varphi_k$ is a quasi-free state satisfying

$$\varphi(a_{i_1}^* \cdots a_{i_r}^* \, a_{j_1} \cdots a_{j_s}) = \delta_{rs} \prod_{l=1}^{s} \delta_{i_l, j_l} \, \mu_{i_l}$$

for all increasing sequences $i_1 < \cdots < i_r$ and $j_1 < \cdots < j_s$. We denote by $\mathcal{W}$ the von Neumann algebra generated by the a_k's in the GNS construction with respect to φ. Then $\mathcal{W}$ is a hyperfinite type III factor and (a_k) are independent with respect to φ.

Example 1.7: Group algebras. Consider a discrete group G. Let $VN(G) \subset B(\ell_2(G))$ be the associated von Neumann algebra generated by the left regular representation $\lambda : G \to B(\ell_2(G))$. More precisely, λ is defined by

$$\big(\lambda(g)f\big)(h) = f(g^{-1}h), \quad f \in \ell_2(G),\ h, g \in G\,,$$

and $VN(G)$ is generated by $\{\lambda(g) : g \in G\}$. Recall that $VN(G)$ is also the w*-closure in $B(\ell_2(G))$ of the algebra of all finite sums $\sum \alpha(g)\lambda(g)$ with $\alpha(g) \in \mathbb{C}$. Let τ_G be the vector state on $VN(G)$ determined by δ_e, where e is the identity of G and $(\delta_g)_{g\in G}$ is the canonical basis of $\ell_2(G)$. τ_G is a normal faithful tracial state on $VN(G)$. If H is a subgroup of G, then $VN(H)$ is identified with the von Neumann subalgebra of $VN(G)$ generated by $\{\lambda(h) : h \in H\}$. The corresponding conditional expectation $\mathcal{E}_{VN(H)}$ is determined by

$$\mathcal{E}_{VN(H)}\big[\sum_{g\in G} \alpha(g)\lambda(g)\big] = \sum_{g\in H} \alpha(g)\lambda(g), \quad \alpha(g) \in \mathbb{C}\,.$$

Now let (G_n) be an increasing sequence of subgroups of G and $g_n \in G_n \setminus G_{n-1}$. Then it is easy to see that $(\lambda(g_n))_n$ is order independent (but not independent in general) with respect to τ_G. In particular, a sequence of free generators on a free group is order independent. Moreover, it is clearly independent.

2. Noncommutative Rosenthal inequality: $p \geq 2$

In this section we prove the noncommutative Rosenthal inequality in the case $p \geq 2$. In this section $\mathcal{M}$ will denote a von Neumann algebra with a normal faithful state φ, and $\mathcal{N} \subset \mathcal{M}$ a φ-invariant von Neumann subalgebra with conditional expectation $\mathcal{E} = \mathcal{E}_{\mathcal{N}}$. Following [JX1], we will also need the diagonal space $\ell_p(L_p(\mathcal{M}))$ whose norm will be denoted by $\|\ \|_{\ell_p(L_p)}$. In the remainder of the paper, c will denote an absolute positive constant which may change from line to line, and c_p a positive constant depending only on p. The notation $A \sim_c B$ will mean that $A \leq c\,B$ and $B \leq c\,A$.

Theorem 2.1: *Let $2 \leq p < \infty$ and $(x_k) \in L_p(\mathcal{M})$ be a finite sequence such that $\mathcal{E}(x_k) = 0$.*

(i) *If (x_k) is independent with respect to $\mathcal{E}$, then*

$$\begin{aligned}\frac{c}{p}\Big\|\sum_k x_k\Big\|_p &\leq \max\Big\{\|(x_k)\|_{\ell_p(L_p)},\, \|(x_k)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^c)},\, \|(x_k)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^r)}\Big\}\\ &\leq 2\Big\|\sum_k x_k\Big\|_p\,.\end{aligned}$$

(ii) *If (x_k) is order independent with respect to $\mathcal{E}$, then*

$$\frac{c}{p^2}\Big\|\sum_k x_k\Big\|_p \le \max\big\{\|(x_k)\|_{\ell_p(L_p)}, \|(x_k)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^c)}, \|(x_k)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^r)}\big\}$$
$$\le 2\Big\|\sum_k x_k\Big\|_p .$$

Proof. (i) Let $(\mathcal{A}_k)$ be a sequence of φ-invariant von Neumann subalgebras of $\mathcal{M}$ which are independent over $\mathcal{N}$ and such that $x_k \in L_p(\mathcal{A}_k)$. Then by Lemma 1.2 (ii) and the fact that $L_p(\mathcal{M})$ is of cotype p with constant 1, we obtain

$$\|(x_k)\|_{\ell_p(L_p)} \le 2\Big\|\sum_k x_k\Big\|_p .$$

On the other hand, by independence,

$$\mathcal{E}(x_k^* x_j) = 0, \quad k \ne j.$$

Thus, for $x = \sum x_k$, we have

$$\Big\|\sum_k \mathcal{E}(x_k^* x_k)\Big\|_{p/2} = \|\mathcal{E}(x^*x)\|_{p/2} \le \|x^*x\|_{p/2} = \|x\|_p^2, .$$

Therefore the lower estimate for the norm of the sum is proved.

The main part is the proof of the upper estimate. First, let us observe that this upper estimate is also true for $1 \le p \le 2$ since

$$\Big\|\sum_k x_k\Big\|_p \le 2\|(x_k)\|_{\ell_p(L_p)} . \tag{2.1}$$

Indeed, this inequality follows immediately from the unconditionality of (x_k) given by Lemma 1.2 (ii) and the type p property of $L_p(\mathcal{M})$. To treat the case $p \ge 2$ we will use a standard iteration procedure. The key step is to show that if the upper estimate is true for some $p \ge 1$, then it is also true for $2p$. This will enable us to iterate, by using (2.1) as a starting point. Thus we assume that for some p there exists a positive constant c_p such that

$$\Big\|\sum_k x_k\Big\|_p \le c_p \max\big\{\|(x_k)\|_{\ell_p(L_p)},\ \|(x_k)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^c)},\ \|(x_k)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^r)}\big\}$$

for all $x_k \in L_p(\mathcal{A}_k)$ with $\mathcal{E}(x_k) = 0$. Our aim is to prove the same estimate for $2p$. Let $x_k \in L_{2p}(\mathcal{A}_k)$ and $\mathcal{E}(x_k) = 0$. First, we apply the noncommutative

Khintchine inequality (cf. [LPP] and also [P1] with the right order of the best constant) and deduce from Lemma 1.2 that

(2.2)

$$\Big\|\sum_k x_k\Big\|_{2p} \le 2\,\mathbb{E}\Big\|\sum_k \varepsilon_k x_k\Big\|_{2p} \le c\sqrt{p}\,\max\Big\{\Big\|\sum_k x_k^* x_k\Big\|_p^{1/2},\ \Big\|\sum_k x_k x_k^*\Big\|_p^{1/2}\Big\},$$

where (ε_k) is a Rademacher sequence and $\mathbb{E}$ denotes the corresponding expectation. Let us consider the first square function on the right hand side. We define the mean zero elements $y_k = x_k^* x_k - \mathcal{E}(x_k^* x_k)$. By assumption, we have

$$\begin{aligned}\Big\|\sum_k x_k^* x_k\Big\|_p &\le \Big\|\sum_k \mathcal{E}(x_k^* x_k)\Big\|_p + \Big\|\sum_k y_k\Big\|_p\\ &\le \Big\|\sum_k \mathcal{E}(x_k^* x_k)\Big\|_p + c_p \max\big\{\|(y_k)\|_{\ell_p(L_p)},\ \|(y_k)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^c)}\big\}\end{aligned}$$

Moreover, if $1 \le p \le 2$, we can disregard the second term in the maximum by virtue of (2.1). Since $\mathcal{E}$ is a contraction on $L_p(\mathcal{M})$, we have

$$\|(y_k)\|_{\ell_p(L_p)} = \Big(\sum_k \Big\|x_k^* x_k - \mathcal{E}(x_k^* x_k)\Big\|_p^p\Big)^{1/p} \le 2\Big(\sum_k \|x_k\|_{2p}^{2p}\Big)^{1/p}.$$

Hence, for $1 \le p \le 2$, we find

$$\begin{aligned}&\Big\|\sum_k x_k\Big\|_{2p}\\ &\quad\le c\sqrt{5p}\,\max\big\{\|(x_k)\|_{\ell_{2p}(L_{2p}(\mathcal{M}))}, \|(x_k)\|_{L_{2p}(\mathcal{M},\mathcal{E};\ell_2^c)}, \|(x_k)\|_{L_{2p}(\mathcal{M},\mathcal{E};\ell_2^r)}\big\}.\end{aligned}$$

Now assume $2 < p < \infty$. We first note that

$$\begin{aligned}\mathcal{E}(y_k^2) &= \mathcal{E}\big[\big(x_k^* x_k - \mathcal{E}(x_k^* x_k)\big)^*\big(x_k^* x_k - \mathcal{E}(x_k^* x_k)\big)\big]\\ &= \mathcal{E}(x_k^* x_k x_k^* x_k) - \mathcal{E}(x_k^* x_k)\mathcal{E}(x_k^* x_k) \le \mathcal{E}(|x_k|^4).\end{aligned}$$

Using [JX1, Lemma 5.2], we obtain

$$\Big\|\sum_k \mathcal{E}(|x_k|^4)\Big\|_{p/2} \le \Big\|\sum_k \mathcal{E}(|x_k|^2)\Big\|_p^{(p-2)/(p-1)}\Big(\sum_k \|x_k\|_{2p}^{2p}\Big)^{1/(p-1)}.$$

By homogeneity, this implies

$$\Big\|\sum_k \mathcal{E}(|x_k|^4)\Big\|_{p/2}^{1/2} \le \max\{\|(x_k)\|^2_{\ell_{2p}(L_{2p})}, \|(x_k)\|^2_{L_{2p}(\mathcal{M},\mathcal{E};\ell_2^c)}\}.$$

Therefore we have proved that

$$\|(y_k)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^c)} \leq \max\left\{\big\|(x_k)\big\|^2_{\ell_{2p}(L_{2p})}, \big\|(x_k)\big\|^2_{L_{2p}(\mathcal{M},\mathcal{E};\ell_2^c)}\right\}.$$

Applying the same arguments to $x_k x_k^*$ and putting together all inequalities so far obtained, we find

$$\Big\|\sum_k x_k\Big\|_{2p}$$
$$\leq c(p(1+2c_p))^{1/2} \max\left\{\|(x_k)\|_{\ell_{2p}(L_{2p})}, \|(x_k)\|_{L_{2p}(\mathcal{M},\mathcal{E};\ell_2^c)}, \|(x_k)\|_{L_{2p}(\mathcal{M},\mathcal{E};\ell_2^r)}\right\}.$$

It thus follows that

$$c_{2p} \leq c(p(1+2c_p))^{1/2}$$

for $p > 2$. We then deduce that $c_{2p} \leq c' 2p$ for some absolute constant c'. Therefore, the induction argument works and we obtain assertion (i).

(ii) The proof of this part is almost the same as the previous one. The only difference is that Lemma 1.2 is no longer at our disposal. In consequence, we have to replace (2.2) by the noncommutative Burkholder-Gundy inequality from [PX1, JX1] (see also [JX2] for the right order of the best constants):

$$\Big\|\sum_k x_k\Big\|_{2p} \leq cp \max\left\{\Big\|\sum_k x_k^* x_k\Big\|_p^{1/2}, \Big\|\sum_k x_k x_k^*\Big\|_p^{1/2}\right\}.$$

This is true for (x_k) is a martingale difference sequence. Indeed, since the von Neumann subalgebra generated by the $\mathcal{A}_k$ is φ-invariant, we may assume that this subalgebra is $\mathcal{M}$ itself. Then letting $\mathcal{M}_k = VN(\mathcal{A}_1, \dots, \mathcal{A}_k)$, we see that $(\mathcal{M}_k)$ is an increasing filtration of subalgebras in the sense of [JX1], which yields a noncommutative martingale structure in $\mathcal{M}$. By Remark 1.1, (x_k) is a martingale difference sequence with respect to $(\mathcal{M}_k)$. The rest of the proof is then the same as that of (i). ∎

Remark 2.2: In the commutative case the best constant in the Rosenthal inequality is of order $p/(1+\log p)$ as $p \to \infty$ (cf. [JSZ]). In view of this result, the constant of order p in the first inequality in Theorem 2.1 seems reasonable. At the time of writing this paper we do not know whether this order is optimal.

Theorem 2.1 deals with independent mean zero variables. For general independent variables, we have the following easy consequence. From now on we will confine our attention only to independence. All subsequent results have counterparts for order independence.

COROLLARY 2.3: *Let p and $\mathcal{M}$ be as in Theorem 2.1. Let $(x_k) \subset L_p(\mathcal{M})$ be an independent sequence with respect to $\mathcal{E}$. Then*

$$\Big\|\sum_k x_k\Big\|_p \le cp\max\Big\{\Big\|\sum_k \mathcal{E}(x_k)\Big\|_p, \|(x_k)\|_{\ell_p(L_p)}, \|(x_k)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^c)}, \|(x_k)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^r)}\Big\}.$$

If additionally all x_k are positive, the inverse inequality holds without constant.

Proof. Let $y_k = x_k - \mathcal{E}(x_k)$. Then

$$\Big\|\sum_k x_k\Big\|_p \le \Big\|\sum_k \mathcal{E}(x_k)\Big\|_p + \Big\|\sum_k y_k\Big\|_p .$$

Now applying Theorem 2.1 to the centered sequence (y_k), we get an equivalence for the second term on the right. Using triangle inequality and $\|\mathcal{E}(x_k)\|_p \le \|x_k\|_p$, we have

$$\|(y_k)\|_{\ell_p(L_p)} \le 2\|(x_k)\|_{\ell_p(L_p)} .$$

For the terms on the conditional square functions, we note that

$$\mathcal{E}(|y_k|^2) = \mathcal{E}(|x_k|^2) - |\mathcal{E}(x_k)|^2 \le \mathcal{E}(|x_k|^2).$$

Then we deduce the desired inequality. To prove the additional part, by the contractivity of $\mathcal{E}$ on $L_p(\mathcal{M})$

$$\Big\|\sum_k x_k\Big\|_p \ge \Big\|\sum_k \mathcal{E}(x_k)\Big\|_p .$$

On the other hand, by Jensen's inequality

$$(2.3)\qquad \Big\|\sum_k |x_k|^2\Big\|_{p/2} = \Big\|\mathbb{E}\Big(\Big|\sum_k \varepsilon_k x_k\Big|^2\Big)\Big\|_{p/2} \le \mathbb{E}\Big\|\sum_k \varepsilon_k x_k\Big\|_p^2 .$$

Note that since $x_k \ge 0$, $-\sum x_k \le \sum \varepsilon_k x_k \le \sum x_k$ for any $\varepsilon_k = \pm 1$; so $\|\sum \varepsilon_k x_k\|_p \le \|\sum x_k\|_p$. Therefore,

$$\|(x_k)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^c)} \le \Big\|\Big(\sum_k |x_k|^2\Big)^{1/2}\Big\|_p \le \Big\|\sum_k x_k\Big\|_p .$$

For the diagonal term, it suffices to note the inequality

$$(2.4)\qquad \|(x_k)\|_{\ell_p(L_p)} \le \Big\|\Big(\sum_k |x_k|^2\Big)^{1/2}\Big\|_p ,$$

which is obtained by interpolating the two cases $p = 2$ and $p = \infty$. Thus the proof of the corollary is complete. ∎

In the case $\mathcal{N} = \mathbb{C}$, our Rosenthal inequality takes a simpler form. Let us formulate this explicitly as follows.

COROLLARY 2.4: *Let $2 \le p < \infty$, and let $(x_k) \subset L_p(\mathcal{M})$ be a sequence independent with respect to φ such that $\mathrm{tr}(x_k D^{1/p'}) = 0$. Then*

$$\Big\|\sum_k x_k\Big\|_p \sim_{cp} \max\Big\{\Big(\sum_k \|x_k\|_p^p\Big)^{1/p}, \Big(\sum_k \mathrm{tr}[(x_k^* x_k + x_k x_k^*)D^{1-2/p}]\Big)^{1/2}\Big\}.$$

In particular, if φ is tracial,

$$\Big\|\sum_k x_k\Big\|_p \sim_{cp} \max\Big\{\Big(\sum_k \|x_k\|_p^p\Big)^{1/p}, \Big(\sum_k \|x_k\|_2^2\Big)^{1/2}\Big\}.$$

Proof. It suffices to observe that for any $q \ge 1$ the conditional expectation $\mathcal{E}_{\mathbb{C}}$ on $L_q(\mathcal{M})$ is given by $\mathcal{E}_{\mathbb{C}}(x) = \mathrm{tr}(xD^{1/q'})D^{1/q}$. ∎

In the same spirit, we have the following Khintchine type inequality.

COROLLARY 2.5: *Keep the assumptions of Corollary 2.4 and assume in addition that*

$$0 < \kappa_1 = \inf_k \mathrm{tr}[(x_k^* x_k + x_k x_k^*)D^{1-2/p}] \quad \textit{and} \quad \sup_k \|x_k\|_p = \kappa_2 < \infty\,.$$

Let $\mathcal{A}$ be another von Neumann algebra, and let $(a_k) \subset L_p(\mathcal{A})$. Then

$$\Big\|\sum_k a_k \otimes x_k\Big\|_{L_p(\mathcal{A}\bar{\otimes}\mathcal{M})} \sim_{c_{p,\kappa_1,\kappa_2}} \Big\|\Big(\sum_k a_k^* a_k + a_k a_k^*\Big)^{1/2}\Big\|_p\,.$$

Proof. We may assume that $\mathcal{A}$ is σ-finite and equipped with a normal faithful state ψ. Then the tensor product $\mathcal{A}\bar{\otimes}\mathcal{M}$ is equipped with $\psi\otimes\varphi$. Identifying $\mathcal{A}$ with a subalgebra of $\mathcal{A}\bar{\otimes}\mathcal{M}$ by $a \leftrightarrow a\otimes 1$, we see that the associated conditional expectation satisfies $\mathcal{E}_{\mathcal{A}}(a\otimes x) = \mathrm{tr}(xD^{1/p'})a$ for $a \in L_p(\mathcal{A})$ and $x \in L_p(\mathcal{M})$. The independence of (x_k) with respect to φ implies that of $(a_k \otimes x_k)$ with respect to $\mathcal{E}_{\mathcal{A}}$. Therefore, by Theorem 2.1, we obtain an equivalence of $\big\|\sum_k a_k \otimes x_k\big\|_p$ with the maximum of three terms. Let us first consider the two terms on the conditional square functions:

$$\begin{aligned}\|(a_k\otimes x_k)\|_{L_p(\mathcal{A}\bar{\otimes}\mathcal{M},\mathcal{E}_{\mathcal{A}};\ell_2^c)} &= \Big\|\Big(\sum_k a_k^*a_k\otimes \mathrm{tr}(x_k^*x_kD^{1-\frac{2}{p}})D^{\frac{2}{p}}\Big)^{1/2}\Big\|_p\\ &= \Big\|\Big(\sum_k \mathrm{tr}(x_k^*x_kD^{1-\frac{2}{p}})\,a_k^*a_k\Big)^{1/2}\Big\|_p\\ &\geq \sqrt{\kappa_1}\Big\|\Big(\sum_k a_k^*a_k\Big)^{1/2}\Big\|_p.\end{aligned}$$

On the other hand, by the Hölder inequality,

$$\mathrm{tr}(x^*x_kD^{1-2/p})\leq \|x_k\|_p^2\leq \kappa_2^2\,.$$

Thus it follows that

$$\|(a_k\otimes x_k)\|_{L_p(\mathcal{A}\bar{\otimes}\mathcal{M},\mathcal{E}_{\mathcal{A}};\ell_2^c)}\sim \Big\|\Big(\sum_k a_k^*a_k\Big)^{1/2}\Big\|_p.$$

Passing to adjoints, we get the same estimate for the other conditional square function. Similarly, we have

$$\|(a_k\otimes x_k)\|_{\ell_p(L_p)}\sim \|(a_k)\|_{\ell_p(L_p)}\,.$$

However, by (2.4)

$$\|(a_k)\|_{\ell_p(L_p)}\leq \Big\|\Big(\sum_k a_k^*a_k\Big)^{1/2}\Big\|_p.$$

Therefore, the assertion follows. ∎

We end this section with a remark on general von Neumann algebras.

Remark 2.6: As stated, our noncommutative Rosenthal inequality holds for σ-finite von Neumann algebras. It can be easily extended to an arbitrary von Neumann algebra $\mathcal{M}$ provided $\mathcal{N}$ and $(\mathcal{A}_k)$ are von Neumann subalgebras of $\mathcal{M}$ such that there exist normal faithful conditional expectations $\mathcal{E}_{\mathcal{N}}:\mathcal{M}\to\mathcal{N}$ and $\mathcal{E}_{\mathcal{A}_k}:\mathcal{M}\to\mathcal{A}_k$ satisfying the commutation relation $\mathcal{E}_{\mathcal{A}_k}\mathcal{E}_{\mathcal{N}}=\mathcal{E}_{\mathcal{N}}\mathcal{E}_{\mathcal{A}_k}=\mathcal{E}_{\mathcal{N}}$. Indeed, let ψ be a strictly normal semifinite faithful weight on $\mathcal{N}$, i.e., a weight of the form $\psi=\sum_{i\in I}\phi_i$, where the ϕ_i are normal states on $\mathcal{N}$ with mutually orthogonal supports. Let e_i be the support of ϕ_i. For a finite subset $J\subset I$, set $e_J = \sum_{i\in J} e_i$. Then (e_J) is an increasing family of projections such that $\lim_J e_J = 1$ strongly. Now we may consider the normal faithful state

$$\varphi_J = \frac{1}{|J|}\sum_{i\in J}\phi_i \circ \mathcal{E}_{\mathcal{N}} \quad \text{on} \quad e_J\mathcal{M}e_J\,.$$

If $(x_k) \subset L_p(\mathcal{M})$ is an independent sequence with respect to $\mathcal{E}_{\mathcal{N}}$ and $(\mathcal{A}_k)$ is the associated independent sequence of subalgebras, we see that the assumptions of Theorem 2.1 are satisfied for $\mathcal{A}_{k,J} = e_J\mathcal{A}_k e_J$. Moreover, for $x \in L_p(\mathcal{M})$ with $p < \infty$ we have

$$x = \lim_J e_J x = \lim_J x e_J = \lim_J e_J x e_J \quad \text{in} \quad L_p(\mathcal{M}).$$

Thus by density, Theorem 2.1 holds in $L_p(\mathcal{M})$. This remark applies to all results proved in this paper. We will not repeat it and consider only the σ-finite case for simplicity.

3. Noncommutative Rosenthal inequality: $p < 2$

We now investigate the noncommutative Rosenthal inequality for $1 < p \leq 2$, which is the dual version of Theorem 2.1. As for the Burkholder inequality in [JX1], this dual version did not exist explicitly in literature even in the commutative (=classical) case. In this section we will assume as before that $\mathcal{N}$ and $(\mathcal{A}_k)$ are φ-invariant von Neumann subalgebras of $\mathcal{M}$ such that $(\mathcal{A}_k)$ is independent with respect to the conditional expectation $\mathcal{E} = \mathcal{E}_{\mathcal{N}}$.

We start by considering the subspace $\mathcal{R}_p^c$ of $L_p(\mathcal{M},\mathcal{E};\ell_2^c)$ consisting of all sequences (x_k) such that $x_k \in L_p(\mathcal{A}_k)$ with $\mathcal{E}(x_k) = 0$, $1 \leq p < \infty$. Alternately, $\mathcal{R}_p^c$ can be defined as the closure in $L_p(\mathcal{M},\mathcal{E};\ell_2^c)$ of all sequences $(a_k D^{1/p})$ such that $a_k \in \mathcal{A}_k$ with $\mathcal{E}(a_k) = 0$. Similarly, we define the corresponding subspaces of $L_p(\mathcal{M},\mathcal{E};\ell_2^r)$ and $\ell_p(L_p(\mathcal{M}))$, which are denoted respectively by $\mathcal{R}_p^r$ and $\mathcal{R}_p^d$.

Lemma 3.1: *Let $1 \leq p < \infty$. Then $\mathcal{R}_p^c$ is 2-complemented in $L_p(\mathcal{M},\mathcal{E};\ell_2^c)$. The similar statements hold for the row and diagonal subspaces $\mathcal{R}_p^r$ and $\mathcal{R}_p^d$.*

Proof. Let us consider a finite sequence $(a_k D^{1/p})$ with $a_k \in \mathcal{M}$. By the Cauchy-Schwarz inequality

$$\mathcal{E}\big(\mathcal{E}_{\mathcal{A}_k}(a_k)^*\mathcal{E}_{\mathcal{A}_k}(a_k)\big) \leq \mathcal{E}\big(\mathcal{E}_{\mathcal{A}_k}(a_k^*a_k)\big) = \mathcal{E}(a_k^*a_k).$$

It follows that

$$\|(\mathcal{E}_{\mathcal{A}_k}(a_k)D^{1/p})\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^c)} \leq \|(a_k D^{1/p})\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^c)}\,.$$

This shows that the map $F((x_k)) = (\mathcal{E}_{\mathcal{A}_k}(x_k))$ defines a contraction on $L_p(\mathcal{M},\mathcal{E};\ell_2^c)$. The same argument shows that $E((x_k)) = (\mathcal{E}(x_k))$ is also a contraction. Then $(\mathrm{id}-E)F$ is the desired projection from $L_p(\mathcal{M},\mathcal{E};\ell_2^c)$ onto $\mathcal{R}_p^c$. This same projection is also bounded from $L_p(\mathcal{M},\mathcal{E};\ell_2^r)$ onto $\mathcal{R}_p^r$ and from $\ell_p(L_p(\mathcal{M}))$ onto $\mathcal{R}_p^d$. ∎

THEOREM 3.2: *Let $1 < p \le 2$. Let $x_k \in L_p(\mathcal{A}_k)$ such that $\mathcal{E}(x_k) = 0$. Then*

$$\frac{1}{2}\Big\|\sum_k x_k\Big\|_p \le \inf_{x_k = x_k^d + x_k^c + x_k^r} \{\|(x_k^d)\|_{\mathcal{R}_p^d} + \|(x_k^c)\|_{\mathcal{R}_p^c} + \|(x_k^r)\|_{\mathcal{R}_p^r}\} \le c\,p'\Big\|\sum_k x_k\Big\|_p .$$

Proof. Let $(x_k) \in \mathcal{R}_p^d$. Then by (2.1),

$$\Big\|\sum_k x_k\Big\|_p \le 2\,\|(x_k)\|_{\mathcal{R}_p^d} .$$

To consider the second term on column norm, let $y_k = a_k D^{1/p}$ with $\mathcal{E}(a_k) = 0$, and set $y = \sum_k y_k$. We deduce from [J1, section 2](see also [JX1, section 7]) that

$$\|y\|_p^2 = \|y^*y\|_{p/2} \le \|\mathcal{E}(y^*y)\|_{p/2} = \Big\|\sum_k D^{1/p}\mathcal{E}(a_k^*a_k)D^{1/p}\Big\|_{p/2} .$$

By density this implies that

$$\Big\|\sum_k x_k\Big\|_p \le \|(x_k)\|_{\mathcal{R}_p^c}$$

whenever $(x_k) \in \mathcal{R}_p^c$. Passing to adjoints, we get the same inequality for the row subspace. Therefore, by triangle inequality we find

$$\Big\|\sum_k x_k\Big\|_p \le 2 \inf_{x_k = x_k^d + x_k^c + x_k^r} \left\{\|(x_k^d)\|_{\mathcal{R}_p^d} + \|(x_k^c)\|_{\mathcal{R}_p^c} + \|(x_k^r)\|_{\mathcal{R}_p^r}\right\} .$$

To prove the converse inequality we use duality. To this end note that the infimum above is the norm of (x_k) in the sum space $\mathcal{R}_p^d + \mathcal{R}_p^c + \mathcal{R}_p^r$. By the duality between sums and intersections, we have

$$(\mathcal{R}_{p'}^d \cap \mathcal{R}_{p'}^c \cap \mathcal{R}_{p'}^r)^* = (\mathcal{R}_{p'}^d)^* + (\mathcal{R}_{p'}^c)^* + (\mathcal{R}_{p'}^r)^*$$

isometrically. However, by Lemma 3.1,

$$(\mathcal{R}_{p'}^d)^* = \mathcal{R}_p^d\,, \quad (\mathcal{R}_{p'}^c)^* = \mathcal{R}_p^c\,, \quad (\mathcal{R}_{p'}^r)^* = \mathcal{R}_p^r$$

isomorphically. Therefore,

$$(\mathcal{R}_{p'}^d \cap \mathcal{R}_{p'}^c \cap \mathcal{R}_{p'}^r)^* = \mathcal{R}_p^d + \mathcal{R}_p^c + \mathcal{R}_p^r .$$

Now let $x_k \in L_p(\mathcal{A}_k)$ with $\mathcal{E}(x_k) = 0$. Let $(y_k) \in \mathcal{R}^d_{p'} \cap \mathcal{R}^c_{p'} \cap \mathcal{R}^r_{p'}$ such that

$$\max\left\{\|(y_k)\|_{\mathcal{R}^d_{p'}}\,,\ \|(y_k)\|_{\mathcal{R}^c_{p'}}\,,\ \|(y_k)\|_{\mathcal{R}^r_{p'}}\right\} \leq 1\,.$$

Then by Theorem 2.1,

$$\Big\|\sum_k y_k\Big\|_{p'} \leq c\,p'\,.$$

Thus, by orthogonality and the Hölder inequality

$$\Big|\sum_k \mathrm{tr}(y_k^* x_k)\Big| = \Big|\mathrm{tr}\Big[\Big(\sum_k y_k\Big)^*\Big(\sum_k x_k\Big)\Big]\Big| \leq c\,p'\,\Big\|\sum_k x_k\Big\|_p\,.$$

We then deduce the desired inequality. Hence the theorem is proved. ∎

Now we give an application to random matrices. Recall that the e_{ij} denote the canonical matrix units of $B(\ell_2)$.

THEOREM 3.3: *Let $1 < p < \infty$ and (x_{ij}) be a finite matrix with entries in $L_p(\mathcal{M})$. Assume that the x_{ij} are independent with respect to $\mathcal{E}$ and $\mathcal{E}(x_{ij}) = 0$. Then for $p \geq 2$*

$$\Big\|\sum_{ij} x_{ij}\otimes e_{ij}\Big\|_{L_p(\mathcal{M}\bar{\otimes}B(\ell_2))} \sim_{cp} \max\Big\{\Big(\sum_{ij}\|x_{ij}\|_p^p\Big)^{1/p}, \Big(\sum_j \Big\|\Big[\sum_i \mathcal{E}(|x_{ij}|^2)\Big]^{1/2}\Big\|_p^p\Big)^{1/p}, \Big(\sum_i \Big\|\Big[\sum_j \mathcal{E}(|x_{ij}{}^*|^2)\Big]^{1/2}\Big\|_p^p\Big)^{1/p}\Big\}$$

and for $p < 2$

$$\Big\|\sum_{ij} x_{ij}\otimes e_{ij}\Big\|_{L_p(\mathcal{M}\bar{\otimes}B(\ell_2))} \sim_{cp'} \inf\Big\{\Big(\sum_{ij}\|x^d_{ij}\|_p^p\Big)^{1/p} + \Big(\sum_j \Big\|\Big[\sum_i \mathcal{E}(|x^c_{ij}|^2)\Big]^{1/2}\Big\|_p^p\Big)^{1/p} + \Big(\sum_i \Big\|\Big[\sum_j \mathcal{E}(|x^r_{ij}{}^*|^2)\Big]^{1/2}\Big\|_p^p\Big)^{1/p}\Big\},$$

where the infimum is taken over all decompositions $x_{ij} = x^d_{ij} + x^c_{ij} + x^r_{ij}$ with mean zero elements x^d_{ij}, x^c_{ij} and x^r_{ij}, which, for each couple (i,j), belong to the von Neumann subalgebra generated by x_{ij}.

Proof. Assume that (x_{ij}) is an $n\times n$ matrix. Let Tr be the usual trace on $B(\ell_2^n)$. Then $\varphi\otimes\mathrm{Tr}$ is a normal faithful positive functional on $\mathcal{M}\bar{\otimes}B(\ell_2^n)$ (which becomes a state if we wish by normalizing Tr). The conditional expectation from $\mathcal{M}\bar{\otimes}B(\ell_2^n)$ onto $\mathcal{N}\bar{\otimes}B(\ell_2^n)$ is $\mathcal{E}\otimes \mathrm{id}_{B(\ell_2^n)}$. It is easy to see that $(x_{ij}\otimes e_{ij})$ is independent with respect to $\mathcal{E}\otimes \mathrm{id}_{B(\ell_2^n)}$. Then the case $p\geq 2$ follows directly from Theorem 2.1. Indeed, we have

$$\begin{aligned}\Big\|\sum_{ij}\mathcal{E}\otimes \mathrm{id}_{B(\ell_2^n)}(|x_{ij}\otimes e_{ij}|^2)\Big\|_{p/2} &= \Big\|\sum_{ij}\mathcal{E}(|x_{ij}|^2)\otimes e_{jj}\Big\|_{p/2}\\ &= \Big(\sum_j\Big\|\sum_i\mathcal{E}(|x_{ij}|^2)\Big\|_{p/2}^{p/2}\Big)^{2/p}.\end{aligned}$$

The same calculation applies to the second square function.

For the case $p<2$ we cannot formally apply Theorem 3.2. However, we can indeed follow the reduction argument of Theorem 3.2 from Theorem 2.1. For this, let $(\mathcal{A}_{ij})$ be a family of subalgebras independent over $\mathcal{N}$ such that $x_{ij}\in L_p(\mathcal{A}_{ij})$. Accordingly, we define $\tilde{\mathcal{R}}_p^c$ to be the subspace of $\ell_p(L_p(\mathcal{M},\mathcal{E};\ell_2^c))$ consisting of (y_{ij}) such that $y_{ij}\in L_p(\mathcal{A}_{ij})$ and $\mathcal{E}(y_{ij})=0$. (Note that ℓ_p and ℓ_2^c in $\ell_p(L_p(\mathcal{M},\mathcal{E};\ell_2^c))$ are in j and i, respectively; this corresponds to the second term in the preceding maximum.) Then the proof of Lemma 3.1 shows that $\tilde{\mathcal{R}}_p^c$ is complemented in $\ell_p(L_p(\mathcal{M},\mathcal{E};\ell_2^c))$. Similarly, we introduce the complemented diagonal and row subspaces $\tilde{\mathcal{R}}_p^d$ and $\tilde{\mathcal{R}}_p^r$ of $\ell_p(\mathbb{N}^2;L_p(\mathcal{M}))$ and $\ell_p(L_p(\mathcal{M},\mathcal{E};\ell_2^r))$, respectively. The rest of the proof is the same as that of Theorem 3.2. ∎

Remark 3.4: Applying Theorem 3.3 to a Rademacher family (ε_{ij}) on a probability space (Ω,μ), we get the following well-known equivalence for $2\leq p<\infty$

$$\begin{aligned}&\Big\|\sum_{ij}\varepsilon_{ij}\,a_{ij}\,e_{ij}\Big\|_{L_p(\Omega;S_p)}\sim\\ &\qquad\max\Big\{\Big(\sum_j\Big(\sum_i|a_{ij}|^2\Big)^{p/2}\Big)^{1/p},\Big(\sum_i\Big(\sum_j|a_{ij}|^2\Big)^{p/2}\Big)^{1/p}\Big\}\end{aligned}$$

for all finite complex matrices (a_{ij}). Indeed, in this special case the diagonal term $\big(\sum_{ij}|a_{ij}|^p\big)^{1/p}$ in the maximum is dominated by each of the two others (see (2.4)). By duality, we get a similar equivalence for $1<p<2$ by replacing, as usual, the maximum by the corresponding infimum (see [LP]). Note that (ε_{ij}) can be replaced by a standard Gaussian family.

Applying the Rosenthal inequality to the independent sequences contained in the examples of Section 1, we get Khintchine type inequalities as in Corollaries 2.4 and 2.5. Due to their importance in applications, we give further details. For convenience, we group them together into two remarks according to the tracial and non-tracial cases.

Remark 3.5: Let φ be a normal faithful tracial state on $\mathcal{M}$, and let (x_k) be a sequence in $L_p(\mathcal{M})$ such that

$$\alpha_p = \inf_k \|x_k\|_p > 0 \quad \text{and} \quad \beta_p = \sup_k \|x_k\|_p < \infty$$

for all $p < \infty$. Assume that the x_k are independent with respect to φ and $\varphi(x_k) = 0$. Let $\mathcal{A}$ be another von Neumann algebra and $(a_k) \subset L_p(\mathcal{A})$ a finite sequence. Then for $2 \le p < \infty$

$$\Big\| \sum_k a_k \otimes x_k \Big\|_{L_p(\mathcal{A}\bar{\otimes}\mathcal{M})} \sim \max\big\{ \|(a_k)\|_{L_p(\mathcal{A};\ell_2^c)}\, , \|(a_k)\|_{L_p(\mathcal{A};\ell_2^r)} \big\}$$

and for $1 < p < 2$

$$\Big\| \sum_k a_k \otimes x_k \Big\|_{L_p(\mathcal{A}\bar{\otimes}\mathcal{M})} \sim \inf\big\{ \|(b_k)\|_{L_p(\mathcal{A};\ell_2^c)} + \|(c_k)\|_{L_p(\mathcal{A};\ell_2^r)} \big\},$$

where the infimum is taken over all decompositions $a_k = b_k + c_k$ in $L_p(\mathcal{A})$. In both cases, the equivalence constants depend only on p, α_p and β_p.

The first equivalence is a special case of Corollary 2.5. The second then follows by duality. This statement implies many known inequalities. For instance, if (x_k) is a Rademacher, Steinhauss or Gaussian sequence, we recover the noncommutative Khintchine inequalities of Lust-Piquard/Pisier [LPP]. As far as for noncommutative independence, (x_k) can be a sequence of free Gaussians, q-Gaussians or free generators. Then we get the corresponding inequalities already in [P1] (except the q-case). It is worth noting that for all these concrete examples, the second equivalence above holds for $p = 1$ also and the constant there is then controlled by a universal one; moreover, in the noncommutative case (except $q \neq -1$) the first equivalence is true even for $p = \infty$ and the constant is also universal (depending only on q in the q-case). We refer to [P1] for more information.

Remark 3.6: Here we consider only the quasi free CAR generators (x_k) defined in (1.2). Then for $2 \le p < \infty$

$$\Big\| \sum_k a_k \otimes D^{1/(2p)} x_k D^{1/(2p)} \Big\|_p \sim$$

$$\max\Big\{ \Big\| \Big(\sum_k (1-\mu_k)^{1/p} \mu_k^{1/p'} a_k^* a_k \Big)^{1/2} \Big\|_p , \Big\| \Big(\sum_k (1-\mu_k)^{1/p'} \mu_k^{1/p} a_k a_k^* \Big)^{1/2} \Big\|_p \Big\}$$

and for $1 < p < 2$

$$\Big\| \sum_k a_k \otimes D^{1/(2p)} x_k D^{1/(2p)} \Big\|_p \sim$$

$$\inf\Big\{ \Big\| (\sum_k (1-\mu_k)^{1/p} \mu_k^{1/p'} b_k^* b_k)^{1/2} \Big\|_p + \Big\| \Big(\sum_k (1-\mu_k)^{1/p'} \mu_k^{1/p} c_k c_k^* \Big)^{1/2} \Big\|_p \Big\},$$

where the infimum is taken over all decompositions $a_k = b_k + c_k$ in $L_p(\mathcal{A})$. Moreover, the equivalence constants depend only on p.

This statement is a reformulation of [X3, Theorem 4.1]. Note that the case $p \ge 2$ can be easily deduced from Corollary 2.5 and the other is again proved by duality. It is shown in [J3] that the second equivalence remains true for $p = 1$. Let us point out that a similar statement holds for the generalized circular system in (1.1). In this case, all constants are universal (see [X2]; see also [JPX] for the q-case). We should emphasize that all these Khintchine type inequalities have interesting applications. In fact, they play a crucial role in the recent works on the operator space Grothendieck theorems and the complete embedding of Pisier's OH into noncommutative L_p, see [J2, PS, X3, X2].

4. A variant using maximal functions

We discuss in this section a version of the noncommutative Rosenthal inequality where the diagonal norm of $\ell_p(L_p(\mathcal{M}))$ is replaced by that of $L_p(\mathcal{M}; \ell_\infty)$. This is in perfect analogy with the classical Burkholder inequality for commutative martingales. Our argument is based on interpolation and the resulting constant presents, unfortunately, a singularity as $p \to 2$. We need some facts on noncommutative $L_p(L_q)$. For our purpose here we will need only the case where the second space L_q is ℓ_q. The investigation of general noncommutative $L_p(L_q)$ spaces will be pursued elsewhere.

Let us recall the definition of the spaces $L_p(\mathcal{M};\ell_\infty)$ and $L_p(\mathcal{M};\ell_1)$, $1 \le p \le \infty$. A sequence (x_k) in $L_p(\mathcal{M})$ belongs to $L_p(\mathcal{M};\ell_\infty)$ if and only if (x_k) admits a factorization $x_k = ay_kb$ with $a, b \in L_{2p}(\mathcal{M})$ and $(y_k) \in \ell_\infty(L_\infty(\mathcal{M}))$. The norm of (x_k) is then defined as

$$\|(x_k)\|_{L_p(\mathcal{M};\ell_\infty)} = \inf_{x_k = ay_kb} \|a\|_{2p} \, \|(y_k)\|_{\ell_\infty(L_\infty)} \, \|b\|_{2p} \, . \tag{4.1}$$

On the other hand, $L_p(\mathcal{M};\ell_1)$ is defined as the space of all sequences $(x_k) \subset L_p(\mathcal{M})$ for which there exist $a_{kj}, b_{kj} \in L_{2p}(\mathcal{M})$ such that

$$x_k = \sum_j a_{kj}^* b_{kj} \, .$$

$L_p(\mathcal{M};\ell_1)$ is equipped with the norm

$$\|(x_k)\|_{L_p(\mathcal{M};\ell_1)} = \inf_{x_k = \sum_j a_{kj}^* b_{kj}} \Big\| \sum_{k,j} a_{kj}^* a_{kj} \Big\|_p^{1/2} \, \Big\| \sum_{k,j} b_{kj}^* b_{kj} \Big\|_p^{1/2} \, .$$

This norm has a description similar to that of $L_p(\mathcal{M};\ell_\infty)$:

$$\|x\|_{L_p(\mathcal{M};\ell_1)} = \inf_{x_k = ay_kb} \|a\|_{2p} \, \|(y_k)\|_{L_\infty(\mathcal{M};\ell_1)} \, \|b\|_{2p} \, . \tag{4.2}$$

We refer to [J1] for more information (see also [JX3]). Now for $1 < q < \infty$ we define $L_p(\mathcal{M};\ell_q)$ as a complex interpolation space between $L_p(\mathcal{M};\ell_\infty)$ and $L_p(\mathcal{M};\ell_1)$:

$$L_p(\mathcal{M};\ell_q) = [L_p(\mathcal{M};\ell_\infty),\ L_p(\mathcal{M};\ell_1)]_{1/q} \, .$$

Our reference for interpolation theory is [BL]. The norm of $L_p(\mathcal{M};\ell_q)$ will be often denoted by $\| \, \|_{L_p(\ell_q)}$. Let us note that if $\mathcal{M}$ is injective, this definition is a special case of Pisier's vector-valued noncommutative L_p-space theory [P1]. The following is also motivated by Pisier's theory.

Proposition 4.1: *Let $(x_k) \subset L_p(\mathcal{M})$. Then $(x_k) \in L_p(\mathcal{M};\ell_q)$ if and only if (x_k) admits a factorization $x_k = ay_kb$ with $a, b \in L_{2p}(\mathcal{M})$ and $(y_k) \in L_\infty(\mathcal{M};\ell_q)$. Moreover,*

$$\|(x_k)\|_{L_p(\ell_q)} = \inf_{x_k = ay_kb} \|a\|_{2p} \, \|(y_k)\|_{L_\infty(\ell_q)} \, \|b\|_{2p} \, .$$

Proof. Let $|||\,(x_k)\,|||_{p,q}$ denote the infimum above. By (4.1) and (4.2), the trilinear map $(a,(y_k),b) \mapsto (ay_kb)$ is contractive from $L_{2p}(\mathcal{M}) \times L_\infty(\mathcal{M};\ell_q) \times L_{2p}(\mathcal{M})$ to $L_p(\mathcal{M};\ell_q)$ for $q = \infty$ and $q = 1$, so is it for any $q \in (1,\infty)$ in virtue of interpolation. This yields

$$\|(x_k)\|_{L_p(\ell_q)} \le |||\,(x_k)\,|||_{p,q} \, .$$

To prove the converse we consider only the case where the state φ is tracial. The general case can be reduced to this one by using Haagerup's reduction theorem as in [X1]. Now assume $\|x\|_{L_p(\ell_q)} < 1$. Let $S = \{z \in \mathbb{C} : 0 \le \mathrm{Re} z \le 1\}$. Then there exists a sequence (f_k) of continuous functions from S to $L_p(\mathcal{M})$, analytic in the interior of S, such that $f_k(1/q) = x_k$ and

$$\sup_{t\in\mathbb{R}} \|(f_k(it))\|_{L_p(\ell_\infty)} \le 1, \quad \sup_{t\in\mathbb{R}} \|(f_k(1+it))\|_{L_p(\ell_1)} \le 1.$$

By (4.1) and (4.2), we have factorizations

$$f_k(z) = a(z)y_k(z)b(z), \quad z \in \partial S$$

such that

$$\|a(z)\|_{2p} \le 1, \quad \|b(z)\|_{2p} \le 1$$

and

$$\|(y_k(it))\|_{L_\infty(\ell_\infty)} \le 1, \quad \|(y_k(1+it))\|_{L_\infty(\ell_1)} \le 1.$$

Moreover, we may assume that a, b and y are strongly measurable on ∂S. Now fix $\varepsilon > 0$. Then by the operator-valued Szegö factorization [PX2, Corollary 8.2], we find two strongly measurable functions $\alpha, \beta : S \to L_{2p}(\mathcal{M})$, analytic in the interior, such that

$$\alpha(z)\alpha(z)^* = a(z)a(z)^* + \varepsilon \quad \text{and} \quad \beta(z)^*\beta(z) = b(z)^*b(z) + \varepsilon\,, \quad z \in \partial S\,.$$

Moreover, $\alpha(z)$ and $\beta(z)$ are invertible for every $z \in S$. For $z \in \partial S$ let $u(z)$ and $v(z)$ be contractions in $\mathcal{M}$ such that

$$a(z) = \alpha(z)u(z) \quad \text{and} \quad b(z) = v(z)\beta(z)\,.$$

We then deduce

$$f_k(z) = \alpha(z)u(z)y_k(z)v(z)\beta(z)\,.$$

Set $\tilde{y}_k(z) = u(z)y_k(z)v(z)$ for $z \in \partial S$. Since $\alpha(z)$ and $\beta(z)$ are invertible, we have $\tilde{y}_k(z) = \alpha(z)^{-1}f_k(z)\beta(z)^{-1}$. Thus $\tilde{y}_k$ is the boundary value of an analytic function in S, so $\tilde{y}_k$ itself may be viewed as an analytic function in S. Therefore, we obtained an analytic factorization of f_k:

$$f_k(z) = \alpha(z)\tilde{y}_k(z)\beta(z), \quad z \in S.$$

Moreover, we have the following estimates

$$\|\alpha(z)\|_{2p} \le 1 + \varepsilon\,, \quad \|\beta(z)\|_{2p} \le 1 + \varepsilon$$

for any $z \in \partial S$ and

$$\|(\tilde{y}_k(it))\|_{L_\infty(\ell_\infty)} \le 1, \quad \|(\tilde{y}_k(1+it))\|_{L_\infty(\ell_1)} \le 1.$$

It then follows that

$$\|\alpha(1/q)\|_{2p} \le 1+\varepsilon\,, \quad \|\beta(1/q)\|_{2p} \le 1+\varepsilon \quad \|(\tilde{y}_k(1/q))\|_{L_\infty(\ell_q)} \le 1.$$

Since $x_k = f_k(1/q) = \alpha(1/q)\tilde{y}_k(1/q)\beta(1/q)$, we deduce

$$\|(x_k)\|_{L_p(\ell_q)} \le 1+\varepsilon\,.$$

Letting $\varepsilon \to 0$ yields $\|(x_k)\|_{L_p(\ell_q)} \le 1$. ∎

Corollary 4.2: *Let* $1 \le p_0, p_1, q_0, q_1 \le \infty$ *and* $0 < \theta < 1$. *Then*

$$[L_{p_0}(\mathcal{M};\ell_{q_0}),\ L_{p_1}(\mathcal{M};\ell_{q_1})]_\theta = L_p(\mathcal{M};\ell_q)$$

with equal norms, where $1/p = (1-\theta)/p_0 + \theta/p_1$ *and* $1/q = (1-\theta)/q_0 + \theta/q_1$.

Proof. By Proposition 4.1, the trilinear map $(a,(y_k),b) \mapsto (ay_kb)$ is contractive from $L_{2p_j}(\mathcal{M}) \times L_\infty(\mathcal{M};\ell_{q_j}) \times L_{2p_j}(\mathcal{M})$ to $L_{p_j}(\mathcal{M};\ell_{q_j})$ for $j=0$ and $j=1$, so by interpolation it is also contractive from $L_{2p}(\mathcal{M}) \times L_\infty(\mathcal{M};\ell_q) \times L_{2p}(\mathcal{M})$ to $[L_{p_0}(\mathcal{M};\ell_{q_0}),\ L_{p_1}(\mathcal{M};\ell_{q_1})]_\theta$. This, together with Proposition 4.1, implies

$$L_p(\mathcal{M};\ell_q) \subset [L_{p_0}(\mathcal{M};\ell_{q_0}),\ L_{p_1}(\mathcal{M};\ell_{q_1})]_\theta\,.$$

The converse inclusion is proved similarly as Proposition 4.1 by using the Szegö factorization. We omit the details. ∎

Corollary 4.3: *Let* $1 \le p, q \le \infty$.

(i) $L_p(\mathcal{M};\ell_p) = \ell_p(L_p(\mathcal{M}))$ *isometrically.*

(ii) *If* $p \le q$,

$$\|(x_k)\|_{L_p(\ell_q)} = \inf_{x_k = ay_kb} \|a\|_{2r}\,\|(y_k)\|_{\ell_q(L_q)}\,\|b\|_{2r}$$

for any $(x_k) \in L_p(\mathcal{M};\ell_q)$, *where* $1/r = 1/p - 1/q$.

(iii) *If* $p \ge q$,

$$\|(x_k)\|_{L_p(\ell_q)} = \sup_{\|\alpha\|_{2s}\le 1,\ \|\beta\|_{2s}\le 1} \|(\alpha x_k \beta)\|_{\ell_q(L_q)}$$

for any $(x_k) \in L_p(\mathcal{M};\ell_q)$, *where* $1/s = 1/q - 1/p$.

Proof. (i) By definition the quality in question is true for $p=\infty$ and $p=1$. For $1<p<\infty$ we use the previous corollary to conclude

$$\begin{aligned} L_p(\mathcal{M};\ell_p) &= [L_\infty(\mathcal{M};\ell_\infty),\ L_1(\mathcal{M};\ell_1)]_{1/p} \\ &= [\ell_\infty(L_\infty(\mathcal{M})),\ \ell_1(L_1(\mathcal{M}))]_{1/p} = \ell_p(L_p(\mathcal{M}))\,. \end{aligned}$$

(ii) Proposition 4.1 may be rewritten symbolically as

$$L_p(\mathcal{M};\ell_q) = L_{2p}(\mathcal{M})\,L_\infty(\mathcal{M};\ell_q)\,L_{2p}(\mathcal{M})\,.$$

However, the Hölder inequality implies

$$L_{2p}(\mathcal{M}) = L_{2r}(\mathcal{M})\,L_{2q}(\mathcal{M}) = L_{2q}(\mathcal{M})\,L_{2r}(\mathcal{M})\,.$$

We thus deduce, by (i)

$$\begin{aligned} L_p(\mathcal{M};\ell_q) &= L_{2r}(\mathcal{M})\,L_{2q}(\mathcal{M})\,L_\infty(\mathcal{M};\ell_q)\,L_{2q}(\mathcal{M})\,L_{2r}(\mathcal{M}) \\ &= L_{2r}(\mathcal{M})\,L_q(\mathcal{M};\ell_q)\,L_{2r}(\mathcal{M}) = L_{2r}(\mathcal{M})\,\ell_q(L_q(\mathcal{M}))\,L_{2r}(\mathcal{M})\,; \end{aligned}$$

whence the desired result.

(iii) Given $(x_k)\in L_p(\mathcal{M};\ell_q)$ we apply Proposition 4.1 to write $x_k = ay_kb$ with $a,b\in L_{2p}(\mathcal{M})$ and $(y_k)\in L_\infty(\mathcal{M};\ell_q)$. Then for any α,β in the unit ball of $L_{2s}(\mathcal{M})$, we have

$$\|(\alpha x_k\beta)\|_{\ell_q(L_q)} \le \|\alpha a\|_{2q}\,\|(y_k)\|_{L_\infty(\ell_q)}\,\|b\beta\|_{2q} \le \|a\|_{2p}\,\|(y_k)\|_{L_\infty(\ell_q)}\,\|b\|_{2p}\,.$$

Therefore,

$$\sup_{\|\alpha\|_{2s}\le 1,\ \|\beta\|_{2s}\le 1} \|(\alpha x_k\beta)\|_{\ell_q(L_q)} \le \|(x_k)\|_{L_p(\ell_q)}\,.$$

To prove the converse inequality, we use (ii) and duality. It suffices to consider a finite sequence $(x_k)_{1\le k\le n}\subset L_p(\mathcal{M})$. Accordingly, we consider the ℓ_q^n-valued L_p-space $L_p(\mathcal{M};\ell_q^n)$. We may also assume $p>q$. Then

$$L_{p'}(\mathcal{M};\ell_1^n)^* = L_p(\mathcal{M};\ell_\infty^n) \quad\text{and}\quad L_{p'}(\mathcal{M};\ell_\infty^n)^* = L_p(\mathcal{M};\ell_1^n)$$

isometrically (see [J2] and [JX3]). Using the duality theorem on complex interpolation, we deduce

$$L_{p'}(\mathcal{M};\ell_{q'}^n)^* = L_p(\mathcal{M};\ell_q^n).$$

Now let $(y_k)\in L_{p'}(\mathcal{M};\ell_{q'}^n)$ be of norm less than 1. By (ii) we can write $y_k = az_kb$ with

$$\|a\|_{2s}\le 1,\quad \|b\|_{2s}\le 1,\quad \|(z_k)\|_{\ell_{q'}(L_{q'})}\le 1.$$

Then

$$\Big|\sum_k \operatorname{tr}(y_k^* x_k)\Big| = \Big|\sum_k \operatorname{tr}(z_k^* a^* x_k b^*)\Big| \le \big\|(a^* x_k b^*)\big\|_{\ell_q(L_q)}\,;$$

whence the desired converse inequality. ∎

Corollary 4.4: *Let $2 \le p \le \infty$. Then*

$$[L_p(\mathcal{M};\ell_2^c),\ L_p(\mathcal{M};\ell_2^r)]_{1/2} \subset L_p(\mathcal{M};\ell_2)\,.$$

Proof. Let $1/r = 1/2 - 1/p$. We consider the map $T : (a,(x_k),b) \mapsto (a x_k b)$. First, we note that

$$T : L_\infty(\mathcal{M}) \times L_p(\mathcal{M};\ell_2^c) \times L_r(\mathcal{M}) \to \ell_2(L_2(\mathcal{M}))$$

is a contraction because

$$\begin{aligned}\sum_k \|a x_k b\|_2^2 \le \|a\|_\infty^2 \sum_k \operatorname{tr}(b^* x_k^* x_k b) &= \|a\|_\infty^2 \operatorname{tr}\Big(\Big(\sum_k x_k^* x_k\Big) b b^*\Big)\\ &\le \|a\|_\infty^2 \Big\|\sum_k x_k^* x_k bigg\|_{p/2} \|bb^*\|_{r/2}\\ &= \|a\|_\infty^2\, \|(x_k)\|_{L_p(\mathcal{M};\ell_2^c)}^2\, \|b\|_r^2\,.\end{aligned}$$

Similarly, we see that

$$T : L_r(\mathcal{M}) \times L_p(\mathcal{M};\ell_2^r) \times L_\infty(\mathcal{M}) \to \ell_2(L_2(\mathcal{M}))$$

is a contraction. Thus by interpolation

$$T : L_{2r}(\mathcal{M}) \times [L_p(\mathcal{M};\ell_2^c),\ L_p(\mathcal{M};\ell_2^r)]_{1/2} \times L_{2r}(\mathcal{M}) \to \ell_2(L_2(\mathcal{M}))$$

is a contraction. Then Corollary 4.3, (iii) implies the assertion. ∎

Remark 4.5: The inclusion converse to that of Corollary 4.4 holds too, so we have equality. This is a special case of the main result from [X1] (see also [JP] for more general results of this type).

Now we are ready to prove the version of the noncommutative Rosenthal inequality in terms of maximal functions.

Theorem 4.6: *Let $\mathcal{N}$ be a φ-invariant von Neumann subalgebra of $\mathcal{M}$ with conditional expectation $\mathcal{E}$. Let $2 < p < \infty$ and $(x_k) \subset L_p(\mathcal{M})$ be a sequence* *independent with respect to $\mathcal{E}$ such that $\mathcal{E}(x_k) = 0$. Then*

$$\Big\| \sum_k x_k \Big\|_p \le c_p \max\big\{ \|(x_k)\|_{L_p(\ell_\infty)}\, , \|(x_k)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^c)}\, , \|(x_k)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^r)} \big\}.$$

Proof. If

$$\|(x_k)\|_{\ell_p(L_p)} < \max\big\{ \|(x_k)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^c)}\, , \|(x_k)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^r)} \big\},$$

then Theorem 2.1 implies

$$\Big\| \sum_k x_k \Big\|_p \le c\, p \max\big\{ \|(x_k)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^c)}\, , \|(x_k)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^r)} \big\}\, ,$$

so we are done. It remains to consider the case where

$$\max\big\{ \|(x_k)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^c)}\, ,\ \|(x_k)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^r)} \big\} \le \|(x_k)\|_{\ell_p(L_p)}\, .$$

Again by Theorem 2.1, we have

$$\Big\| \sum_k x_k \Big\|_p \le c\, p \|(x_k)\|_{\ell_p(L_p)}\, .$$

By the reiteration theorem, we deduce (with $\theta = 2/p$)

$$L_p(\mathcal{M};\ell_p) = [L_p(\mathcal{M};\ell_\infty),\ L_p(\mathcal{M};\ell_2)]_\theta\, .$$

This, together with Corollary 4.3 (i), implies

$$\big\|(x_k)\big\|_{\ell_p(L_p)} \le \big\|(x_k)\big\|_{L_p(\ell_\infty)}^{1-\theta} \, \big\|(x_k)\big\|_{L_p(\ell_2)}^{\theta}\, .$$

Using Lemma 1.2 and (2.3), we have

$$\max\{ \|(x_k)\|_{L_p(\mathcal{M};\ell_2^c)}\, ,\ \|(x_k)\|_{L_p(\mathcal{M};\ell_2^r)} \} \le 2 \Big\| \sum_k x_k \Big\|_p\, . \tag{4.3}$$

Then by Corollary 4.4

$$\|(x_k)\|_{L_p(\ell_2)} \le 2 \Big\| \sum_k x_k \Big\|_p\, .$$

Combining these estimates we find (after cancellation) that

$$\Big\| \sum_k x_k \Big\|_p \le (c\, 2^\theta p)^{1/(1-\theta)} \, \|(x_k)\|_{L_p(\ell_\infty)}\, .$$

The theorem is thus proved with $c_p \le (c' p)^{p/(p-2)}$ for $p > 2$. In particular, $c_p \le c'' p$ for $p \ge 4$. ∎

We take this opportunity to present the same improvement in the context of the noncommutative Burkholder inequality of [JX1]. Namely, we want to replace the norm $\|(dx)\|_{\ell_p(L_p)}$ in the following inequality by $\|(dx)\|_{L_p(\ell_\infty)}$:

$$\|x\|_p \le c_p \max\left\{\|(dx)\|_{\ell_p(L_p)}\,, \|x\|_{h_p^c}\,, \|x\|_{h_p^r}\right\}$$

for any noncommutative martingale $x = (x_k)$ with respect to an increasing filtration $(\mathcal{E}_k)$ of normal faithful conditional expectations. Here $dx = (dx_k)$ denotes the difference sequence of x and

$$\|x\|_{h_p^c} = \left\|\left(\sum_k \mathcal{E}_{k-1}(|dx_k|^2)\right)^{1/2}\right\|_p, \quad \|x\|_{h_p^r} = \|x^*\|_{h_p^c}\,.$$

We refer to [JX1] for more details. Note that $c_p \le c\,p$ according to [R3], which improves the original estimate $c_p \le c\,p^2$ from [JX1].

Theorem 4.7: *Let $2 < p < \infty$. Then for any noncommutative bounded L_p-martingale x we have*

$$\|x\|_p \le c_p' \max\left\{\|(dx)\|_{L_p(\ell_\infty)}\,, \|x\|_{h_p^c}\,, \|x\|_{h_p^r}\right\}.$$

Proof. This proof is almost the same as that of the previous theorem. The only difference is that the martingale analogue of (4.3) is now obtained by using the lower estimate in the noncommutative Burkholder-Gundy inequality (see [JX2] for the optimal order of the constant):

$$\max\{\|(dx)\|_{L_p(\mathcal{M};\ell_2^c)}\,, \|(dx)\|_{L_p(\mathcal{M};\ell_2^r)}\} \le c\,p\,\|x\|_p\,.$$

We omit the details. The resulting order of the constant c_p' is the same as that of c_p in the previous theorem. ∎

Remark 4.8: We can also improve the lower estimates in the noncommutative Burkholder/Rosenthal inequalities for $1 < p < 2$, by replacing the diagonal term $\ell_p(L_p)$ by $L_p(\ell_1)$. For instance, under the assumptions of Theorem 3.2 we have

$$\inf_{x_k = x_k^d + x_k^c + x_k^r} \left\{\|(x_k^d)\|_{\tilde{\mathcal{R}}_p^d} + \|(x_k^c)\|_{\mathcal{R}_p^c} + \|(x_k^r)\|_{\mathcal{R}_p^r}\right\} \le c_p \left\|\sum_k x_k\right\|_p,$$

where $\tilde{\mathcal{R}}_p^d$ is the subspace of $L_p(\mathcal{M};\ell_1)$ consisting of all (x_k) such that $x_k \in L_p(\mathcal{A}_k)$ with $\mathcal{E}(x_k) = 0$. The proof is similar to that of Theorem 3.2 via duality. The complementation of the space $\tilde{\mathcal{R}}_p^d$ follows from the noncommutative Doob inequality in [J1].

5. The nonfaithful case

Nonfaithful filtrations of von Neumann subalgebras, therefore nonfaithful conditional expectations, occur very naturally in operator algebra theory. The simplest example is the natural filtration $(\mathbb{M}_n)_{n\geq 1}$ of $B(\ell_2)$ given by the algebras $\mathbb{M}_n$ of matrices (a_{ij}) such that $a_{ij}=0$ if $\max(i,j)>n$. On the other hand, the notion of nonfaithful copies in a tensor product of von Neumann algebras is important in the context of iterated ultraproducts of von Neumann algebras.

The aim of this section is to extend Theorem 2.1 to the case of nonfaithful conditional expectations. We start with the relevant notion. $\mathcal{M}$ is still assumed to be σ-finite and equipped with a normal faithful state φ. Let $\mathcal{N}$ be a w*-closed involutive (not necessarily unital) subalgebra of $\mathcal{M}$. Let e be the unit of $\mathcal{N}$, so e is a projection of $\mathcal{M}$. Again, assume that $\mathcal{N}$ is φ-invariant (i.e., $\sigma_t^{\varphi}(\mathcal{N})\subset\mathcal{N}$ for all $t\in\mathbb{R}$). With these assumptions we still have a normal conditional expectation $\mathcal{E}_{\mathcal{N}}:\mathcal{M}\to\mathcal{N}$ with support equal to e such that $\varphi\circ\mathcal{E}_{\mathcal{N}}=\varphi_e$, where $\varphi_e=e\varphi e$. Like in the faithful case, $\mathcal{E}_{\mathcal{N}}$ extends to a contractive projection from $L_p(\mathcal{M})$ onto $L_p(\mathcal{N})$ for every $p\geq 1$. We refer to [JX1] for more details.

Now, we consider a sequence $(\mathcal{A}_k)$ of φ-invariant w*-closed involutive subalgebras of $\mathcal{M}$ containing $\mathcal{N}$. Let us denote by r_k the unit of $\mathcal{A}_k$. We will say that the algebras $\mathcal{A}_k$ are **independent over $\mathcal{N}$** or **with respect to $\mathcal{E}_{\mathcal{N}}$** if

(i) the projections $s_k=r_k-e$ are mutually orthogonal;
(ii) for every k, $\mathcal{E}_{\mathcal{N}}(xy)=\mathcal{E}_{\mathcal{N}}(x)\mathcal{E}_{\mathcal{N}}(y)$ holds for all $x\in\mathcal{A}_k$ and y in the w*-closed involutive subalgebra generated by $(\mathcal{A}_j)_{j\neq k}$.

Note that in this case $(e\mathcal{A}_k e)$ is faithfully independent over $\mathcal{N}$ in the sense of Section 1. A sequence $(x_k)\subset L_p(\mathcal{M})$ is called **independent with respect to $\mathcal{E}_{\mathcal{N}}$** if there exists a sequence $(\mathcal{A}_k)$ of subalgebras independent with respect to $\mathcal{E}_{\mathcal{N}}$ such that $x_k\in L_p(\mathcal{A}_k)$.

The new ingredient for the nonfaithful version of the noncommutative Rosenthal inequality is a separate treatment of the corners. In the rest of this section we will assume that $(\mathcal{A}_k)$ is independent with respect to $\mathcal{E}=\mathcal{E}_{\mathcal{N}}$ and keep the preceding notations.

Lemma 5.1: *Let $2\leq p<\infty$ and $x_k\in L_p(\mathcal{A}_k)$. Then*

$$\Big\|\sum_k s_k x_k e\Big\|_p\leq c\sqrt{p}\,\max\big\{\|(s_kx_ke)\|_{\ell_p(L_p)},\ \|(s_kx_k)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^c)}\big\}.$$

Proof. Let $x = \sum_k s_k x_k e$. By the orthogonality of the s_k, we obtain

$$\begin{aligned}\|x\|_p^2 =& \Big\| \sum_k e x_k^* s_k x_k e \Big\|_{p/2} \\ \le& \Big\| \sum_k \mathcal{E}(x_k^* s_k x_k) \Big\|_{p/2} + \Big\| \sum_k e x_k^* s_k x_k e - \mathcal{E}(x_k^* s_k x_k) \Big\|_{p/2}.\end{aligned}$$

Note that $y_k = e x_k^* s_k x_k e - \mathcal{E}(x_k^* s_k x_k) \in e\mathcal{A}_k e$ and satisfies $\mathcal{E}(y_k) = 0$. As observed before, the sequence $(e\mathcal{A}_k e)$ is faithfully independent over $\mathcal{N}$. Now, we follow the proof of Theorem 2.1. If $2 \le p \le 4$, we deduce from (2.1) that

$$\Big\| \sum_k y_k \Big\|_{p/2} \le 2\mathbb{E} \Big\| \sum_k \varepsilon_k y_k \Big\|_{p/2} \le 2\Big(\sum_k \|y_k\|_{p/2}^{p/2} \Big)^{2/p} \le 4\Big(\sum_k \|s_k x_k e\|_p^p \Big)^{2/p}.$$

For $4 < p < \infty$, we deduce from Theorem 2.1 applied to $(y_k) \subset L_q(e\mathcal{M}e)$ with $q = p/2$ and Lemma [JX1, Lemma 5.2] that

$$\begin{aligned}\Big\| \sum_k y_k \Big\|_q \le& cp \max\Big\{ \|(y_k)\|_{\ell_q(L_q)}, \Big\| \Big(\sum_k \mathcal{E}(y_k^* y_k) \Big)^{1/2} \Big\|_q \Big\} \\ \le& cp \max\Big\{ \|(s_k x_k e)\|_{\ell_p(L_p)}^2, \Big\| \sum_k \mathcal{E}(|s_k x_k e|^4) \Big\|_{p/4}^{1/2} \Big\} \\ \le& cp \max\Big\{ \|(s_k x_k e)\|_{\ell_p(L_p)}^2, \Big(\sum_k \|s_k x_k e\|_p^p \Big)^{1/(p-2)} \\ & \Big\| \sum_k \mathcal{E}(|s_k x_k e|^2) \Big\|_{p/2}^{(p/2-2)/(p-2)} \Big\}.\end{aligned}$$

Then the assertion follows by homogeneity. ∎

The nonfaithful version of the Rosenthal inequality for $p \ge 2$ has the same form as Theorem 2.1.

THEOREM 5.2: *Let $2 \le p < \infty$ and $x_k \in L_p(A_k)$. Set $y_k = x_k - \mathcal{E}(x_k)$. Then*

$$\Big\| \sum_k x_k \Big\|_p \sim_{cp} \max\Big\{ \Big\| \sum_k \mathcal{E}(x_k) \Big\|_p, \|(y_k)\|_{\ell_p(L_p)}, \|(y_k)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^c)}, \|(y_k)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^r)} \Big\}.$$

Proof. Since

$$\Big\| \sum_k x_k \|_p \le \Big\| \sum_k \mathcal{E}(x_k) \Big\|_p + \Big\| \sum_k y_k \Big\|_p ,$$

we need only to estimate the second term on the right. Since y_k is supported by r_k and $s_k = r_k - e$ for each k, we have

$$\Big\| \sum_k y_k \Big\|_p \le \Big\| \sum_k s_k y_k s_k \Big\|_p + \Big\| \sum_k s_k y_k e \Big\|_p + \Big\| \sum_k e y_k s_k \Big\|_p + \Big\| \sum_k e y_k e \Big\|_p .$$

By the mutual orthogonality of the s_k,

$$\Big\| \sum_k s_k y_k s_k \Big\|_p = \Big(\sum_k \|s_k y_k s_k\|_p^p \Big)^{1/p} \le \|(y_k)\|_{\ell_p(L_p)} .$$

On the other hand, by Lemma 5.1,

$$\begin{aligned} \Big\| \sum_k s_k y_k e \Big\|_p &\le c\sqrt{p} \max\big\{ \|(s_k y_k e)\|_{\ell_p(L_p)},\ \|(s_k y_k)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^c)} \big\} \\ &\le c\sqrt{p} \max\big\{ \|(y_k)\|_{\ell_p(L_p)},\ \|(y_k)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^c)} \big\}. \end{aligned}$$

Passing to adjoints, we get the same estimate for another term on the corners. To deal with the last term, we recall that the algebras $e\mathcal{A}_k e$ are faithfully independent over $\mathcal{N}$. Thus Theorem (2.1) applies to $(e y_k e)$:

$$\begin{aligned} \Big\| \sum_k e y_k e \Big\|_p &\le cp \max\big\{ \|(e y_k e)\|_{\ell_p(L_p)}, \|(e y_k e)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^c)}, \|(e y_k e)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^r)} \big\} \\ &\le cp \max\big\{ \|(y_k)\|_{\ell_p(L_p)}, \|(y_k)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^c)}, \|(y_k)\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^r)} \big\}. \end{aligned}$$

Combining the preceding inequalities, we obtain the upper estimate. The lower estimate is proved in the same way as in the faithful case. ∎

Example 5.3: Nonfaithful independence occurs naturally in the context of conditional expectations with respect to corners. Let $\mathcal{M}$ be a von Neumann algebra, e a projection and (r_k) a family of projections such that $e \le r_j$ and such that the s_k are mutually orthogonal, where $s_k = r_k - e$. Consider

$$\mathcal{N} = e\mathcal{M}e \quad \text{and} \quad \mathcal{A}_k = r_k \mathcal{M} r_k .$$

The conditional expectation associated with $\mathcal{N}$ is given by $\mathcal{E}(x) = exe$. Then the $\mathcal{A}_k$ are independent over $\mathcal{N}$. This situation occurs for example on a tensor product $\mathcal{M} = \mathcal{B}^{\otimes n}$, where $e = f_1 \otimes \cdots \otimes f_n$ and $r_k = f_1 \otimes \cdots f_{k-1} \otimes 1 \otimes f_{k+1} \otimes \cdots \otimes f_n$ with f_k projections in $\mathcal{B}$.

Remark 5.4: There exists, of course, a nonfaithful version of the Rosenthal inequality for $1 < p \leq 2$. We keep the same assumptions as before. The main technical difference is that we have to introduce two extra spaces

$$R_p(se) = \Big\{ \sum_k s_k x_k e \ : \ x_k \in L_p(\mathcal{A}_k),\ \mathcal{E}(x_k) = 0 \Big\}$$

and

$$R_p(es) = \Big\{ \sum_k e x_k s_k \ : \ x_k \in L_p(\mathcal{A}_k),\ \mathcal{E}(x_k) = 0 \Big\}.$$

It is easy to show that they are complemented in

$$\Big\{ \sum_k x_k \ : \ x_k \in L_p(\mathcal{A}_k),\ \mathcal{E}(x_k) = 0 \Big\}.$$

Thanks to Lemma 5.1, we are able to describe the dual $R_{p'}(se)$ of $R_p(se)$ as an intersection of two terms, an $\ell_{p'}$-term and a column square function. Using the duality argument from the proof of Theorem 3.2, we deduce

$$\Big\| \sum_k s_k x_k e \Big\|_p \sim_{c\sqrt{p'}} \inf_{s_k x_k e = s_k x_k^d e + s_k x_k^c e} \big\| (s_k x_k^d e) \big\|_{\ell_p(L_p)} + \big\| (s_k x_k^c) \big\|_{L_p(\mathcal{M},\mathcal{E};\ell_2^c)} .$$

A similar result holds for $R_p(es)$. Now let $x_k \in L_p(A_k)$ with $\mathcal{E}(x_k) = 0$. Then

$$\Big\| \sum_k x_k \Big\|_p \sim_c$$
$$\max \Big\{ \big\| (s_k x_k s_k) \big\|_{\ell_p(L_p)}, \ \Big\| \sum_k s_k x_k e \Big\|_p, \ \Big\| \sum_k e x_k s_k \Big\|_p, \ \Big\| \sum_k e x_k e \Big\|_p \Big\}.$$

The second and third terms were already treated. However, the last term is the faithful part, so can be dealt with according to Theorem 3.2, which yields an equivalence with an infimum. This complicated expression involving maximum and infimum is particularly interesting in connection with independent copies (as in [J3]). In this case, the expressions are symmetric. This formula can be used to prove that subsymmetric sequences in $L_p(\mathcal{M})$, $1 < p \leq 2$, are symmetric (see [JR] for more details).

6. Symmetric subspaces of noncommutative L_p

In this section, we present some applications of the Burkholder/Rosenthal inequalities to the study of symmetric subspaces of noncommutative L_p-spaces both in the category of Banach spaces and in that of operator spaces. The results obtained are the noncommutative or operator space analogues of the corresponding results in [JMST]. Thus we will follow arguments in [JMST] in many cases. It will be convenient to state these results in parallel for both categories, which will also ease comparing and understanding them. All unexplained Banach space terminologies used in the sequel can be found in [LT]. We refer to [ER, P2] for background on operator spaces and completely bounded maps and to [P1, JNRX] for the operator space structure of noncommutative L_p-spaces. In this paper we will focus on subspaces of these spaces. In this situation we will only need the following fact from [P1]: If X and Y are subspaces of $L_p(\mathcal{M})$, $1 \le p \le \infty$, then the cb-norm of a linear map $T : X \to Y$ is given by

$$\|T\|_{cb} = \|\mathrm{id}_{S_p} \otimes T : S_p(X) \to S_p(Y)\| .$$

Here $S_p(X)$ denotes the closure of $S_p \otimes X$ in $L_p(B(\ell_2)\bar{\otimes}\mathcal{M})$. In other words, the cb-norm is calculated with matrix-valued coefficients instead of scalar-valued coefficients for the usual norm $\|T\|$. It is then straightforward to transfer to this setting all Banach space terminologies concerning bases, basic sequences, etc. For instance, a basic sequence $(x_k) \subset X$ is said to be completely unconditional if there exists a constant λ such that

$$\Big\| \sum_k \varepsilon_k a_k \otimes x_k \Big\| \le \lambda \Big\| \sum_k a_k \otimes x_k \Big\|$$

for all $a_k \in S_p$ and $\varepsilon_k = \pm 1$. Similarly, a FDD (finite dimensional decomposition) (F_k) of X is said to be completely unconditional if there exists a constant λ such that

$$\Big\| \sum_k \varepsilon_k a_k \otimes x_k \Big\| \le \lambda \Big\| \sum_k a_k \otimes x_k \Big\|$$

for all $x_k \in F_k$, $a_k \in S_p$ and $\varepsilon_k = \pm 1$.

The von Neumann algebras considered in this section and the next one may be non σ-finite. However, since we will often consider sequences or separable subspaces in $L_p(\mathcal{M})$, it is easy to bring $\mathcal{M}$ to a σ-finite subalgebra (see also Remark 2.6).

Lemma 6.1: *Let $\mathcal{M}$ be a hyperfinite type III_λ factor with $0 \le \lambda \le 1$ and with separable predual. Let $1 < p < \infty$. Then $L_p(\mathcal{M})$ has a completely unconditional FDD.*

Proof. In the range $0 < \lambda \le 1$, we may assume that $\mathcal{M}$ is an ITPFI factor. In general (including $\lambda = 0$), we can always find a normal faithful state φ and an increasing sequence of finite dimensional φ-invariant subalgebras $\mathcal{M}_n$ with conditional expectations $\mathcal{E}_n : \mathcal{M} \to \mathcal{M}_n$ (see [JRX]). This yields a martingale structure on $\mathcal{M}$. We define the difference operators $\mathcal{D}_n = \mathcal{E}_n - \mathcal{E}_{n-1}$ where $\mathcal{E}_0 = 0$. Note that the spaces $F_n = \mathcal{D}_n(L_p(\mathcal{M}))$ are finite dimensional and every element can be written uniquely as $x = \sum_n \mathcal{D}_n(x)$. Thus $L_p(\mathcal{M})$ has a FDD. The complete unconditionality of this decomposition means that all maps $T_\varepsilon = \sum_n \varepsilon_n \mathcal{D}_n$ are completely bounded uniformly in $\varepsilon_n = \pm 1$. Namely, the maps $\mathrm{id}_{S_p} \otimes T_\varepsilon$ are uniformly bounded, i.e., there exists a constant c such that

$$\Big\| \sum_n \varepsilon_n (\mathrm{id}_{S_p} \otimes \mathcal{D}_n)(x) \Big\|_p \le c \Big\| \sum_n (\mathrm{id}_{S_p} \otimes \mathcal{D}_n)(x) \Big\|_p$$

holds for all choices of signs (ε_n) and $x \in L_p(B(\ell_2)\bar{\otimes}\mathcal{M})$. But this inequality is a direct consequence of the noncommutative Burkholder-Gundy inequalities [PX1, JX1]. Moreover, the constant c depends only on p. ∎

THEOREM 6.2: *Let $\mathcal{M}$ be a hyperfinite von Neumann algebra. Let $2 < p < \infty$, and let $(x_n) \subset L_p(\mathcal{M})$ be a sequence of unit vectors, which converges weakly to 0. Then there exist constants $0 \le \alpha, \beta \le 1$, depending only on (x_n), and a subsequence $(\tilde{x}_n)$ of (x_n) such that*

$$\Big\| \sum_n a_n \otimes \tilde{x}_n \Big\|_p \sim_{c_p} \max\Big\{ \Big(\sum_n \|a_n\|_p^p \Big)^{1/p}, \alpha \Big\| \Big(\sum_n a_n^* a_n \Big)^{1/2} \Big\|_p, \beta \Big\| \Big(\sum_n a_n a_n^* \Big)^{1/2} \Big\|_p \Big\}$$

holds for all finite sequences $(a_n) \subset S_p$.

Proof. The first part of the proof is to show that we can reduce our problem to the case where $L_p(\mathcal{M})$ has a completely unconditional FDD. To this end we first use a standard procedure to reduce $\mathcal{M}$ to a von Neumann algebra with separable predual (see [GGMS, Appendix]). Indeed, assume that $\mathcal{M}$ is σ-finite and let φ be a normal faithful state on $\mathcal{M}$. Let $A \subset \mathcal{M}$ be a countable subset, and let $\mathcal{M}_A$ be the von Neumann subalgebra generated by $\sigma_t^\varphi(a)$ with $a \in A$ and $t \in \mathbb{Q}$. Then $\mathcal{M}_A$ has separable predual. Moreover, $\mathcal{M}_A$ is φ-invariant. Consequently, there is a normal faithful conditional expectation from $\mathcal{M}$ onto $\mathcal{M}_A$, thus $L_p(\mathcal{M}_A)$ is a complemented subspace of $L_p(\mathcal{M})$. Now writing each

x_n as a convergent series of elements from $\mathcal{M}D^{1/p}$: $x_n = \sum_k a_{nk}D^{1/p}$, we can take $\{a_{nk} : n, k \in \mathbb{N}\}$ as A. Then $x_n \in L_p(\mathcal{M}_A)$. Therefore, replacing $\mathcal{M}$ by $\mathcal{M}_A$, we may assume that $\mathcal{M}_*$ is separable.

Now if $\mathcal{M}$ is semifinite, then by [P1, Theorem 3.4] $\mathcal{M}$ has an increasing filtration of finite dimensional subalgebras; so as in the proof of Lemma 6.1, we deduce that $L_p(\mathcal{M})$ has a completely unconditional FDD. To treat the case where $\mathcal{M}$ is of type III, we use another standard trick in order to ensure that we may work with a factor. To this end, we consider the crossed product $\mathcal{R} = \overline{\bigotimes}_{n\in\mathbb{N}}(\mathcal{M}, \varphi) \rtimes G$ between the infinite tensor product $\overline{\bigotimes}_{n\in\mathbb{N}}(\mathcal{M}, \varphi)$ and the discrete group G of all finite permutations on $\mathbb{N}$. Any finite permutation acts on the infinite tensor product by shuffling the corresponding coordinates. Clearly, we also have a normal faithful conditional expectation $\mathcal{E} : \mathcal{R} \to \mathcal{M}$ obtained by first projecting onto the identity element of G and then to the first component in the infinite tensor product. This implies that $L_p(\mathcal{M})$ can be identified as a (complemented) subspace of $L_p(\mathcal{R})$. On the other hand, according to [HW, Proof of Theorem 2.6], $\mathcal{R}$ is a hyperfinite factor. Thus $\mathcal{R}$ is of type III$_\lambda$ for some $0 \le \lambda \le 1$ (see [C, H2]). Therefore, Lemma 6.1 implies that $L_p(\mathcal{R})$ has a completely unconditional FDD given by a filtration $(\mathcal{E}_k)$ of normal faithful conditional expectations. In the remainder of the proof, replacing $\mathcal{M}$ by $\mathcal{R}$ if necessary, we may assume that $L_p(\mathcal{M})$ itself has this FDD.

The second part of the proof follows very closely its commutative model (see [JMST, Theorem 1.14]). Using the gliding hump procedure, we may find a perturbation of a subsequence $(\hat{x}_n)$ and a corresponding subsequence $(\hat{\mathcal{E}}_k)$ such that

(i) $\hat{\mathcal{E}}_n(\hat{x}_n) = \hat{x}_n$;
(ii) $\hat{\mathcal{E}}_n(\hat{x}_k) = 0$ for all $k > n$;
(iii) $\lim_k \hat{\mathcal{E}}_n(\hat{x}_k^*\hat{x}_k) = y_n$ and $\|\hat{\mathcal{E}}_n(\hat{x}_k^*\hat{x}_k) - y_n\|_{p/2} \le \varepsilon 2^{-k}$ for $k > n$;
(iv) $\lim_k \hat{\mathcal{E}}_n(\hat{x}_k\hat{x}_k^*) = z_n$ and $\|\hat{\mathcal{E}}_n(\hat{x}_k\hat{x}_k^*) - z_n\|_{p/2} \le \varepsilon 2^{-k}$ for $k > n$.

Here $\varepsilon > 0$ is arbitrarily given and will be chosen after knowing the y_n's. It follows immediately from (iii) that (y_n) is a bounded $L_{p/2}$-martingale with respect to $(\hat{\mathcal{E}}_n)$. Since $p/2 > 1$, (y_n) converges to some $y \in L_{p/2}(\mathcal{M})$. Similarly, we obtain that (z_n) converges to some $z \in L_{p/2}(\mathcal{M})$. We define $\alpha = \|y\|_{p/2}^{1/2}$ and $\beta = \|z\|_{p/2}^{1/2}$. Passing to subsequences of $(\hat{x}_n)$ and $(\hat{\mathcal{E}}_k)$ if necessary, we may further assume

$$\big\|\hat{\mathcal{E}}_{n-1}(\hat{x}_n^*\hat{x}_n)-y\big\|_{p/2} \le 2^{-(n+1)}\|y\|_{p/2}\,, \quad \big\|\hat{\mathcal{E}}_{n-1}(\hat{x}_n\hat{x}_n^*)-z\big\|_{p/2} \le 2^{-(n+1)}\|z\|_{p/2}\,.$$

Note that (i) and (ii) imply that $(\hat{x}_n)$ is a martingale difference sequence with respect to $(\hat{\mathcal{E}}_n)$. Thus applying the noncommutative Burkholder inequality [JX1], we find, for any $a_n \in S_p$,

$$\Big\|\sum_n a_n \otimes \hat{x}_n\Big\|_p \sim_{c_p} \Big(\sum_n \|a_n \otimes \hat{x}_n\|_p^p\Big)^{1/p} + \Big\|\sum_n a_n^* a_n \otimes \hat{\mathcal{E}}_{n-1}(\hat{x}_n^*\hat{x}_n)\Big\|_{p/2}^{1/2} + \Big\|\sum_n a_n a_n^* \otimes \hat{\mathcal{E}}_{n-1}(\hat{x}_n\hat{x}_n^*)\Big\|_{p/2}^{1/2}.$$

From perturbation, we have $1/2 \le \|\hat{x}_n\|_p \le 2$, so the first diagonal term on the right is fine. On the other hand, the triangle inequality implies

$$\begin{aligned}\Big\|\sum_n a_n^* a_n \otimes \hat{\mathcal{E}}_{n-1}(\hat{x}_n^*\hat{x}_n) - \sum_n a_n^* a_n \otimes y\Big\|_{p/2} &\le \sum_n \|a_n^* a_n\| \|\hat{\mathcal{E}}_{n-1}(\hat{x}_n^*\hat{x}_n) - y\|_{p/2} \\ &\le \frac{1}{2}\|y\|_{p/2} \sup_n \|a_n^* a_n\|_{p/2} \le \frac{\alpha^2}{2}\Big\|\sum_n a_n^* a_n\Big\|_{p/2}.\end{aligned}$$

It follows that

$$\Big|\Big\|\sum_n a_n^* a_n \otimes \hat{\mathcal{E}}_{n-1}(\hat{x}_n^*\hat{x}_n)\Big\|_{p/2} - \Big\|\sum_n a_n^* a_n \otimes y\Big\|_{p/2}\Big| \le \frac{\alpha^2}{2}\Big\|\sum_n a_n^* a_n\Big\|_{p/2}.$$

However,

$$\Big\|\sum_n a_n^* a_n \otimes y\Big\|_{p/2} = \alpha^2 \Big\|\sum_n a_n^* a_n\Big\|_{p/2}.$$

Therefore, we deduce

$$\Big\|\sum_n a_n^* a_n \otimes \hat{\mathcal{E}}_{n-1}(\hat{x}_n^*\hat{x}_n)\Big\|_{p/2} \sim_c \alpha^2 \Big\|\sum_n a_n^* a_n\Big\|_{p/2}.$$

The same argument applies to the last term on the row norm. Keeping in mind that $(\hat{x}_n)$ is a perturbation of a subsequence $(\tilde{x}_n)$ of (x_n) and going back to this subsequence, we get the announced result. ∎

As a first application, we present an operator space version of the Kadec-Pełzsyński alternative. For this we need some notation from the theory of operator spaces. The spaces C_p and R_p are defined as the column and row subspaces of S_p, respectively. Namely,

$$C_p = \overline{\text{span}}\,\{e_{k1} \ : \ k \in \mathbb{N}\} \quad \text{and} \quad R_p = \overline{\text{span}}\,\{e_{1k} \ : \ k \in \mathbb{N}\}\,.$$

Note that as Banach spaces, C_p and R_p are isometric to ℓ_2 by identifying both (e_{k1}) and (e_{1k}) with the canonical basis (e_k) of ℓ_2. We will adopt this identification in the sequel. This permits us to consider the intersection $C_p \cap R_p$. Recall that the operator space structures of these spaces are determined as follows. For any finite sequence $(a_k) \subset S_p$,

$$\Big\| \sum_k a_k \otimes e_k \Big\|_{S_p(C_p)} = \Big\| \Big(\sum_k a_k^* a_k \Big)^{1/2} \Big\|_p ,$$
$$\Big\| \sum_k a_k \otimes e_k \Big\|_{S_p(R_p)} = \Big\| \Big(\sum_k a_k a_k^* \Big)^{1/2} \Big\|_p$$

and

$$\Big\| \sum_k a_k \otimes e_k \Big\|_{S_p(C_p \cap R_p)} = \max \Big\{ \Big\| \Big(\sum_k a_k^* a_k \Big)^{1/2} \Big\|_p , \Big\| \Big(\sum_k a_k a_k^* \Big)^{1/2} \Big\|_p \Big\}.$$

Recall that a sequence (x_k) in a Banach space X is said to be semi-normalized if $\inf_k \|x_k\| > 0$ and $\sup_k \|x_k\| < \infty$.

Corollary 6.3: *Assume that $\mathcal{M}$ is hyperfinite. Let $2 \le p < \infty$ and $(x_n) \subset L_p(\mathcal{M})$ be a semi-normalized sequence which converges to 0 weakly. Then (x_n) contains a subsequence $(\tilde{x}_n)$ which is completely equivalent to the canonical basis of ℓ_p, C_p, R_p or $C_p \cap R_p$.*

Proof. Assume $p > 2$. Let $(\tilde{x}_n)$ be the subsequence from Theorem 6.2. If $\alpha = \beta = 0$, then $(\tilde{x}_n)$ is completely equivalent to the basis of ℓ_p. If $\alpha > 0$ and $\beta = 0$, then we find a copy of C_p by virtue of (2.4). Similarly, if $\alpha = 0$ and $\beta > 0$ it turns out to be R_p. The case $\alpha > 0$ and $\beta > 0$ yields $C_p \cap R_p$. ∎

A basis (x_k) of $X \subset L_p(\mathcal{M})$ is called symmetric if there exists a positive constant λ such that

$$\Big\| \sum_k \varepsilon_k \alpha_{\pi(k)} x_k \Big\|_p \le \lambda \Big\| \sum_k \alpha_k x_k \Big\|_p$$

holds for finite sequences $(\alpha_k) \subset \mathbb{C}$, $\varepsilon_k = \pm 1$ and permutations π of the positive integers. In this case, X is called a symmetric space. The least constant λ (over all possible symmetric bases of X) is denoted by $\mathrm{sym}(X)$. Again, we transfer this definition to the operator space setting: (x_k) is completely symmetric if

$$\Big\| \sum_k \varepsilon_k a_{\pi(k)} \otimes x_k \Big\|_p \le \lambda \Big\| \sum_k a_k \otimes x_k \Big\|_p$$

holds for finite sequences $(a_k) \subset S_p$, $\varepsilon_k = \pm 1$ and permutations π. If X is a completely symmetric space, the relevant constant is denoted by $\mathrm{sym}_{cb}(X)$. It is clear that the four spaces in the previous corollary are completely symmetric. Thus we deduce the following

Corollary 6.4: *Let $\mathcal{M}$ and p be as above. Then every infinite dimensional subspace of $L_p(\mathcal{M})$ contains an infinite completely symmetric basic sequence.*

It is not known whether the assertion above holds for $1 \le p < 2$. This problem is open even for scalar coefficients. On the other hand, we also do not know whether the hyperfiniteness assumption can be removed for $2 < p < \infty$. We refer to [RX] and [R1] for different versions of the Kadec-Pełczyński alternative, which are most often at the Banach space level.

We now show that conversely all completely symmetric subspaces of noncommutative L_p are only those found in Corollary 6.3. The next result is our starting point.

Theorem 6.5: *Let $\mathcal{M}$ be a von Neumann algebra, $2 \le p < \infty$ and $x_{ij} \in L_p(\mathcal{M})$. Then*

$$\Big(\mathbb{E} \Big\| \sum_{i=1}^n \varepsilon_i x_{i\,\pi(i)} \Big\|_p^p \Big)^{1/p} \sim_{c_p}$$
$$\max \Big\{ \Big(\frac{1}{n} \sum_{i,j=1}^n \|x_{ij}\|_p^p \Big)^{1/p}, \Big\| \Big(\frac{1}{n} \sum_{i,j=1}^n (x_{ij}^* x_{ij} + x_{ij} x_{ij}^*) \Big)^{1/2} \Big\|_p \Big\}.$$

Here the expectation $\mathbb{E}$ is taken over all choices of signs $\varepsilon_i = \pm 1$ and all permutations π on $\{1, \ldots, n\}$.

Proof. Again, we can assume that $\mathcal{M}$ is equipped with a normal faithful state φ. We consider $\Omega = \{-1,1\}^n \times \Pi_n$, where Π_n is the set of all permutations on $\{1, \ldots, n\}$. The Haar measure on this group is the product measure $\mu = \varepsilon \otimes \nu$ of the normalized counting measures ε and ν on $\{-1,1\}^n$, Π_n, respectively. The underlying von Neumann algebra is then given by $(\mathcal{N}, \psi) =$

$L_\infty(\Omega, 2^\Omega, \mu) \otimes (\mathcal{M}, \varphi)$. In order to apply the noncommutative Burkholder inequality we have to use the right filtration taken from [JMST]. For $k = 1, \ldots, n$ we consider the functions $f_k : \Pi_n \to \mathbb{R}$, $f_k(\pi) = \pi(k)$. The σ-algebra Σ_k^2 is defined as the smallest σ-algebra on Π_n making $f_1, \ldots, f_k$ measurable. By Σ_k^1 we denote the smallest σ-algebra on $\{-1,1\}^n$ making $\varepsilon_1, \ldots, \varepsilon_k$ measurable, where $\varepsilon_1, \ldots, \varepsilon_n$ are the coordinate functions on $\{-1,1\}^n$. Let Σ_k be the product σ-algebra $\Sigma_k^1 \times \Sigma_k^2$. We then define the filtration $(\mathcal{N}_k)_k$ of ψ-invariant subalgebras by

$$\mathcal{N}_k = L_\infty(\Omega, \Sigma_k, \mu) \otimes \mathcal{M} \,.$$

Let $\mathbb{E}_k$ be the conditional expectation associated to Σ_k. Then $\mathcal{E}_k = \mathbb{E}_k \otimes \mathrm{id}$ is the state preserving conditional expectation from $\mathcal{N}$ onto $\mathcal{N}_k$.

After these preliminaries, we consider

$$x = \sum_{i=1}^n \varepsilon_i x_{i\,\pi(i)} \in L_p(\mathcal{N}).$$

Let $d_k = dx_k$ be the martingale differences of x with respect to $(\mathcal{N}_k)$. We note that $d_k = \varepsilon_k x_{k\,\pi(k)}$. Therefore, the noncommutative Burkholder inequality [JX1] implies

$$\|x\|_p \le c_p \max\Big\{ \Big(\sum_{k=1}^n \|x_{k\,\pi(k)}\|_p^p \Big)^{1/p} , \Big\| \sum_{k=1}^n \mathcal{E}_{k-1}(x_{k\,\pi(k)}^* x_{k\,\pi(k)} + x_{k\,\pi(k)} x_{k\,\pi(k)}^*) \Big\|_{p/2}^{1/2} \Big\}.$$

Clearly, for every $k = 1, \ldots, n$, we have

$$\begin{aligned} \|x_{k\,\pi(k)}\|_p^p &= \sum_{j=1}^n \nu(\{\pi : \pi(k) = j\})\, \|x_{kj}\|_p^p \\ &= \sum_{j=1}^n \frac{(n-1)!}{n!} \|x_{kj}\|_p^p = \frac{1}{n} \sum_{j=1}^n \|x_{kj}\|_p^p \,. \end{aligned}$$

Hence,

$$\Big(\sum_{k=1}^n \|x_{k\,\pi(k)}\|_p^p \Big)^{1/p} = \Big(\frac{1}{n} \sum_{k,j=1}^n \|x_{kj}\|_p^p \Big)^{1/p} .$$

Let $\mathbb{E}_k^2$ be the conditional expectation onto $L_\infty(\Pi_n, \Sigma_k^2, \mu)$. We observe that

$$\mathcal{E}_{k-1}(x_{k\,\pi(k)}^* x_{k\,\pi(k)}) = (\mathbb{E}_{k-1}^2 \otimes \mathrm{id})(x_{k\,\pi(k)}^* x_{k\,\pi(k)}) \,.$$

The atoms in Σ^2_{k-1} are indexed by $(k-1)$-tuples $(i_1,\dots,i_{k-1})$ of distinct integers in $\{1,\dots,n\}$. More precisely,

$$A_{(i_1,\dots,i_{k-1})} = \{\pi \,:\, \pi(1)=i_1,\dots,\pi(k-1)=i_{k-1}\}.$$

Clearly, the cardinality of $A_{(i_1,\dots,i_{k-1})}$ is that of $\Pi_{n-(k-1)}$, i.e., $(n-k+1)!$. Therefore, letting $\alpha_k = (n-k+1)!/n!$, we get

$$\begin{aligned}(\mathbb{E}^2_{k-1}\otimes \mathrm{id})(x^*_{k\,\pi(k)}x_{k\,\pi(k)}) \\ = \sum_{(i_1,\dots,i_{k-1})} 1\!\!1_{A_{(i_1,\dots,i_{k-1})}}\alpha_k^{-1}\int_{A_{(i_1,\dots,i_{k-1})}} x^*_{k\,\pi(k)}x_{k\,\pi(k)}\,d\nu(\pi)\,.\end{aligned}$$

For fixed $(i_1,\dots,i_{k-1})$, letting $B=\{i_1,\dots,i_{k-1}\}$, we have

$$\alpha_k^{-1}\int_{A_{(i_1,\dots,i_{k-1})}} x^*_{k\,\pi(k)}x_{k\,\pi(k)}\,d\nu(\pi) = \frac{1}{n-k+1}\sum_{j\notin B} x^*_{kj}x_{kj}\,.$$

Hence for all $k\le n/2$ we deduce

$$(\mathbb{E}^2_{k-1}\otimes \mathrm{id})(x^*_{k\,\pi(k)}x_{k\,\pi(k)}) \le \frac{2}{n}\sum_{j=1}^n x^*_{kj}x_{kj}\,.$$

Let us assume temporarily that $x_{kj}=0$ for $k>n/2$. Then combining the previous estimates, we obtain

$$\sum_{k=1}^n \mathcal{E}_{k-1}(x^*_{k\,\pi(k)}x_{k\,\pi(k)}) \le \frac{2}{n}\sum_{k,j=1}^n x^*_{kj}x_{kj}$$

for all permutations π. The same argument applies to $x_{k\,\pi(k)}x^*_{k\,\pi(k)}$ too. Therefore, we get the upper estimate under the additional assumption that $x_{kj}=0$ for $k>n/2$. The general case then follows from triangle inequality.

For the lower estimate we use the Jensen inequality and the orthogonality of the Rademacher variables (noting that $p/2\ge 1$):

$$\begin{aligned}\|x\|_p^2 &= \mathbb{E}\,\|x^*x\|_{p/2} \ge \|\mathbb{E}(x^*x)\|_{p/2} \\ &= \Big\|\sum_{k=1}^n\int_{\Pi_n} x^*_{k\,\pi(k)}x_{k\,\pi(k)}\Big\|_{p/2} = \Big\|\frac{1}{n}\sum_{k,j=1}^n x^*_{kj}x_{kj}\Big\|_{p/2}\,.\end{aligned}$$

The same calculation involving xx^* yields the other square function estimate. Since $L_p(\mathcal{N})$ has cotype p, we easily find the missing estimate on the diagonal term. ∎

COROLLARY 6.6: *Let $\mathcal{A}$ and $\mathcal{M}$ be von Neumann algebras and $2 \le p < \infty$. Let $(x_k)_{1\le k\le n} \subset L_p(\mathcal{M})$ and $\lambda > 0$ such that*

$$\Big\| \sum_{k=1}^n \varepsilon_k a_{\pi(k)} \otimes x_k \Big\|_p \le \lambda \Big\| \sum_{k=1}^n a_k \otimes x_k \Big\|_p$$

holds for all $\varepsilon_k = \pm 1$, all permutations π on $\{1, \ldots, n\}$ and coefficients $a_k \in L_p(\mathcal{A})$. Then there are constants α, β and γ, depending only on (x_k), such that

$$\Big\| \sum_{k=1}^n a_k \otimes x_k \Big\|_p \sim_{\lambda^2 c_p} \max\Big\{ \alpha \Big(\sum_{k=1}^n \|a_k\|_p^p \Big)^{1/p}, \\ \beta \Big\| \Big(\sum_{k=1}^n a_k^* a_k \Big)^{1/2} \Big\|_p, \gamma \Big\| \Big(\sum_{k=1}^n a_k a_k^* \Big)^{1/2} \Big\|_p \Big\}$$

holds for all $a_k \in L_p(\mathcal{A})$.

Proof. This is an easy consequence of Theorem 6.5. Indeed, we have

$$\frac{1}{\lambda} \Big\| \sum_{k=1}^n a_k \otimes x_k \Big\|_p \le \Big(\mathbb{E} \Big\| \sum_{k=1}^n \varepsilon_k a_{\pi(k)} \otimes x_k \Big\|_p^p \Big)^{1/p} \le \lambda \Big\| \sum_{k=1}^n a_k \otimes x_k \Big\|_p .$$

Then we deduce the assertion with

$$\alpha = \Big(\frac{1}{n} \sum_{k=1}^n \|x_k\|_p^p \Big)^{1/p}, \quad \beta = \Big\| \Big(\frac{1}{n} \sum_{k=1}^n x_k^* x_k \Big)^{1/2} \Big\|_p,$$
$$\gamma = \Big\| \Big(\frac{1}{n} \sum_{k=1}^n x_k x_k^* \Big)^{1/2} \Big\|_p . \qquad \blacksquare$$

Let us introduce a more notation. For a Banach (or operator) space X and positive real α, αX denotes X but equipped with the norm $\alpha\| \, \|$. For convenience, set $\alpha X = \{0\}$ if $\alpha = 0$. Recall that the Banach-Mazur distance between two Banach spaces X and Y is

$$d(X, Y) = \inf \big\{ \|T\| \, \|T^{-1}\| : \ T : E \to F \text{ isomorphism} \big\}.$$

Similarly, we define the operator space analogue $d_{cb}(X, Y)$ by replacing the norm of an isomorphism by the cb-norm of a complete isomorphism.

COROLLARY 6.7: *Let $2 \le p < \infty$ and X be an n-dimensional subspace of $L_p(\mathcal{M})$. Then*

(i) *there exist nonnegative* α *and* β *such that*

$$d(X,\ \alpha\ell_p^n \cap \beta\ell_2^n) \leq c_p \,\mathrm{sym}(X)^2\,;$$

(ii) *there exist nonnegative* α, β *and* γ *such that*

$$d_{cb}(X,\ \alpha\ell_p^n \cap \beta C_p^n \cap \gamma R_p^n) \leq c_p \,\mathrm{sym}_{cb}(X)^2\,.$$

Proof. Let $(x_1, \ldots, x_n)$ be a (completely) symmetric basis of X with constant $\lambda \leq 2\,\mathrm{sym}(X)$ (or $\lambda \leq 2\,\mathrm{sym}_{cb}(X)$). Such a basis exists for $\dim X < \infty$. It then remains to apply the previous corollary with $\mathcal{A} = \mathbb{C}$ for (i) and $\mathcal{A} = B(\ell_2)$ for (ii). ∎

Corollary 6.8: *Let* $2 \leq p < \infty$ *and* $X \subset L_p(\mathcal{M})$ *be an infinite dimensional subspace.*

(i) *If* X *is symmetric, then* X *is isomorphic to* ℓ_p *or* ℓ_2.
(ii) *If* X *is completely symmetric, then* X *is completely isomorphic to* ℓ_p, C_p, R_p *or* $C_p \cap R_p$.

Proof. We prove only (ii). Then the proof of (i) is simpler; it is done just by replacing vector coefficients by scalar ones. Let (x_k) be a completely symmetric basis of X with constant λ. For every $n \in \mathbb{N}$, set

$$\alpha_n = \Big(\frac{1}{n}\sum_{k=1}^n \|x_k\|_p^p\Big)^{1/p}, \beta_n = \Big\|\Big(\frac{1}{n}\sum_{k=1}^n x_k^* x_k\Big)^{1/2}\Big\|_p, \gamma_n = \Big\|\Big(\frac{1}{n}\sum_{k=1}^n x_k x_k^*\Big)^{1/2}\Big\|_p.$$

Note that α_n, β_n and γ_n are less than or equal to $\sup_k \|x_k\|_p$. Passing to subsequences if necessary, we may assume that the three sequences (α_n), (β_n) and (γ_n) converge respectively to α, β and γ. Thus by Corollary 6.6, for any finite sequence $(a_k) \subset S_p$ we have

$$\Big\|\sum_k a_k \otimes x_k\Big\|_p \sim_{\lambda^2 c_p} \max\Big\{\alpha\Big(\sum_k \|a_k\|_p^p\Big)^{1/p}, \\ \beta\Big\|\Big(\sum_k a_k^* a_k\Big)^{1/2}\Big\|_p,\ \gamma\Big\|\Big(\sum_k a_k a_k^*\Big)^{1/2}\Big\|_p\Big\}.$$

Then using (2.4), we deduce that X is completely isomorphic to ℓ_p if $\beta = \gamma = 0$, to C_p if $\beta > 0$ and $\gamma = 0$, to R_p if $\beta = 0$ and $\gamma > 0$, and finally to $R_p \cap C_p$ if $\beta > 0$ and $\gamma > 0$. ∎

Remark 6.9: It will be shown in [JR] that every subsymmetric basic sequence in $L_p(\mathcal{M})$ is symmetric. A sequence (e_k) is called subsymmetric if

$$\Big\| \sum_k \varepsilon_k a_k \otimes e_{j_k} \Big\|_p \sim_c \Big\| \sum_k a_k \otimes e_k \Big\|$$

holds for all increasing sequences (j_k) of integers. Therefore, Corollary 6.8 yields a characterization of subspaces of noncommutative L_p with a subsymmetric basis.

If $\mathcal{M}$ is finite, we can eliminate the two spaces C_p and R_p in Corollary 6.8 (ii).

Corollary 6.10: *Let $2 < p < \infty$ and $\mathcal{M}$ be a finite von Neumann algebra. Then C_p and R_p do not completely embed into $L_p(\mathcal{M})$.*

Proof. We assume that φ is a normal faithful tracial state on $\mathcal{M}$. Suppose that C_p completely embeds into $L_p(\mathcal{M})$. Namely, there exists an infinite sequence $(x_k) \subset L_p(\mathcal{M})$ such that

$$\Big\| \sum_k a_k \otimes x_k \Big\|_p \sim \Big\| \Big(\sum_k a_k^* a_k \Big)^{1/2} \Big\|_p$$

holds for all $(a_k) \subset S_p$. In particular, if $\alpha = (\alpha_{ik}) \in S_p$, then

$$\|\alpha\|_{S_p} \sim \Big\| \sum_{i,k} \alpha_{ik} e_{1\,i} \otimes x_k \Big\|_{L_p(B(\ell_2)\bar{\otimes}\mathcal{M})} .$$

Note that for

$$x = \sum_{i,k} \alpha_{ik} e_{1\,i} \otimes x_k \quad \text{and} \quad x_i = \sum_k \alpha_{ik} x_k \,,$$

the Hölder inequality implies

$$\begin{aligned} \|x\|^2_{L_2(B(\ell_2)\bar{\otimes}\mathcal{M})} &= \|xx^*\|_{L_1(B(\ell_2)\bar{\otimes}\mathcal{M})} = \Big\| \sum_i x_i x_i^* \Big\|_{L_1(\mathcal{M})} \\ &\le \Big\| \sum_i x_i x_i^* \Big\|_{L_{p/2}(\mathcal{M})} = \|x\|^2_{L_p(B(\ell_2)\bar{\otimes}\mathcal{M})} . \end{aligned}$$

Here the $L_p(\mathcal{M})$ are defined in terms of the trace φ. This tells us that on the subspace $Y = \mathrm{span}\{e_{1\,i} \otimes x_k\}$ the norms in $L_p \cap L_2$ and L_p coincide. Thus we have found an embedding of S_p into $L_p(B(\ell_2)\bar{\otimes}\mathcal{M}) \cap L_2(B(\ell_2)\bar{\otimes}\mathcal{M})$. According to [J4] the latter space embeds into $L_p(\mathcal{R})$ for a finite von Neumann algebra $\mathcal{R}$.

Thus we obtain an embedding of S_p into $L_p(\mathcal{R})$. This is, however, absurd in view of the results in [Su]. ∎

7. Bisymmetric and unitary invariant subspaces of L_p

We extend in this section the results in the previous one to the case of double indices. Namely, we will determine the bisymmetric and unitary invariant subspaces of noncommutative L_p-spaces for $2 < p < \infty$. In particular, we will characterize those unitary ideals which can embed into a noncommutative L_p. For notational convenience, given a finite matrix $x = (x_{ij})$ with entries in $L_p(\mathcal{M})$ we introduce

$$\gamma_0(x) = \Big(\sum_{i,j} \|x_{ij}\|_p^p\Big)^{1/p},$$

$$\gamma_1(x) = \Big(\sum_i \Big\|\Big(\sum_j x_{ij}^* x_{ij}\Big)^{1/2}\Big\|_p^p\Big)^{1/p}, \ \gamma_2(x) = \Big(\sum_i \Big\|\Big(\sum_j x_{ij} x_{ij}^*\Big)^{1/2}\Big\|_p^p\Big)^{1/p},$$

$$\gamma_3(x) = \Big(\sum_j \Big\|\Big(\sum_i x_{ij}^* x_{ij}\Big)^{1/2}\Big\|_p^p\Big)^{1/p}, \ \gamma_4(x) = \Big(\sum_j \Big\|\Big(\sum_i x_{ij} x_{ij}^*\Big)^{1/2}\Big\|_p^p\Big)^{1/p},$$

$$\gamma_5(x) = \Big\|\Big(\sum_{i,j} x_{ij}^* x_{ij}\Big)^{1/2}\Big\|_p, \qquad \gamma_6(x) = \Big\|\Big(\sum_{i,j} x_{ij} x_{ij}^*\Big)^{1/2}\Big\|_p,$$

$$\gamma_7(x) = \Big\|\sum_{i,j} e_{ij} \otimes x_{ij}\Big\|_{L_p(B(\ell_2)\bar{\otimes}\mathcal{M}))}, \quad \gamma_8(x) = \Big\|\sum_{i,j} e_{ji} \otimes x_{ij}\Big\|_{L_p(B(\ell_2)\bar{\otimes}\mathcal{M}))}.$$

THEOREM 7.1: *Let $2 \le p < \infty$ and $\mathcal{A}$ and $\mathcal{M}$ be von Neumann algebras. Let $a = (a_{ij})$ and $x = (x_{ij})$ be two $n \times n$ matrices with entries in $L_p(\mathcal{A})$ and $L_p(\mathcal{M})$, respectively. Then*

$$\Big(\mathbb{E}\Big\|\sum_{i,j=1}^n \varepsilon_i \varepsilon_j'\, a_{ij} \otimes x_{\pi(i)\,\pi'(j)}\Big\|_p\Big)^{1/p} \sim_{c_p}$$
$$\frac{1}{n^{2/p}}\,\gamma_0(a)\gamma_0(x) + \frac{1}{n^{1/p+1/2}}\sum_{k=1}^4 \gamma_k(a)\gamma_k(x) + \frac{1}{n}\sum_{k=5}^8 \gamma_k(a)\gamma_k(x)\,.$$

Here the expectation $\mathbb{E}$ is taken over independent copies ε_i, ε_i' of Rademacher variables and independent copies π and π' of permutations on $\{1, \ldots, n\}$.

Proof. This is an iteration of Theorem 6.5. By that theorem, we get

$$\Big(\mathbb{E}\Big\|\sum_{i,j=1}^{n}\varepsilon_i\varepsilon_j' a_{ij}\otimes x_{\pi(i)\pi'(j)}\Big\|_p\Big)^{1/p}\sim$$
$$\frac{1}{n^{1/p}}\Big(\mathbb{E}_{\varepsilon',\pi'}\sum_{i,k}\Big\|\sum_j \varepsilon_j' a_{ij}\otimes x_{k\,\pi'(j)}\Big\|_p^p\Big)^{1/p}$$
$$+\frac{1}{n^{1/2}}\Big(\mathbb{E}_{\varepsilon',\pi'}\Big\|\sum_{i,k,j}\varepsilon_j'\, e_{(i,k),(1,1)}\otimes a_{ij}\otimes x_{k\,\pi'(j)}\Big\|_p^p\Big)^{1/p}$$
$$+\frac{1}{n^{1/2}}\Big(\mathbb{E}_{\varepsilon',\pi'}\Big\|\sum_{i,k,j}\varepsilon_j'\, e_{(1,1),(i,k)}\otimes a_{ij}\otimes x_{k\,\pi'(j)}\Big\|_p^p\Big)^{1/p}\overset{\text{def}}{=} I+II+III\,.$$

Here we use $e_{(i,k),(j,l)}$ to denote the matrix units of $B(\ell_2(\mathbb{N}^2))$, so (i,k) and (j,l) index rows and columns, respectively. We apply Theorem 6.5 for a second time to the first term on the right hand side and find

$$I\sim\frac{1}{n^{2/p}}\,\gamma_0(a)\gamma_0(x)+\frac{1}{n^{1/p+1/2}}\Big(\sum_{i,k}\Big\|\sum_{j,l}e_{(j,l),(1,1)}\otimes a_{ij}\otimes x_{kl}\Big\|_p^p\Big)^{1/p}$$
$$+\ \frac{1}{n^{1/p+1/2}}\Big(\sum_{i,k}\Big\|\sum_{j,l}e_{(1,1),(j,l)}\otimes a_{ij}\otimes x_{kl}\Big\|_p^p\Big)^{1/p}.$$

Identifying $e_{(j,l),(1,1)}$ with $e_{j1}\otimes e_{l1}$ (up to a conjugation by a unitary), we have

$$\Big\|\sum_{j,l}e_{(j,l),(1,1)}\otimes a_{ij}\otimes x_{kl}\Big\|_p=\Big\|\Big(\sum_j e_{j1}\otimes a_{ij}\Big)\otimes\Big(\sum_l e_{l1}\otimes x_{kl}\Big)\Big\|_p$$
$$=\Big\|\Big(\sum_j a_{ij}^*a_{ij}\Big)^{1/2}\Big\|_p\Big\|\Big(\sum_l x_{kl}^*x_{kl}\Big)^{1/2}\Big\|_p.$$

We deal with similarly the other term containing $e_{(1,1),(j,l)}$ and then deduce that

$$I\sim\frac{1}{n^{2/p}}\,\gamma_0(a)\gamma_0(x)+\frac{1}{n^{1/p+1/2}}\,\gamma_1(a)\gamma_1(x)+\frac{1}{n^{1/p+1/2}}\,\gamma_2(a)\gamma_2(x)\,.$$

Similar arguments apply to II and III too. II is again equivalent to a sum of three terms. Let us consider, for instance, the second one on column norm, which is

$$\frac{1}{n}\Big\| \sum_{j,l}\sum_{i,k} e_{(j,l),(1,1)} \otimes e_{(i,k),(1,1)} \otimes a_{ij} \otimes x_{kl} \Big\|_p$$
$$= \frac{1}{n}\Big\| \sum_{i,j} e_{(i,j),(1,1)} \otimes a_{ij} \Big\|_p \Big\| \sum_{k,l} e_{(k,l),(1,1)} \otimes x_{kl} \Big\|_p$$
$$= \frac{1}{n}\,\gamma_5(a)\gamma_5(x)\,.$$

Then we see that

$$II \sim \frac{1}{n^{1/p+1/2}}\,\gamma_3(a)\gamma_3(x) + \frac{1}{n}\,\gamma_5(a)\gamma_5(x) + \frac{1}{n}\,\gamma_7(a)\gamma_7(x)\,.$$

Finally, III yields the three missing terms. ∎

Permutations and $(\varepsilon_1, \ldots, \varepsilon_n)$ induce permutation and diagonal matrices, which are, of course, unitary. If the expectation in Theorem 7.1 is taken over all unitary matrices, we get a much simpler equivalence.

Theorem 7.2: *Under the assumption of Theorem 7.1, we have*

$$\Big(\mathbb{E}\Big\| \sum_{i,j,k,l=1}^{n} u_{ik}v_{lj}\, a_{ij} \otimes x_{kl} \Big\|_p\Big)^{1/p} \sim_{c_p} \frac{1}{n}\sum_{k=5}^{8} \gamma_k(a)\gamma_k(x)\,.$$

Here the expectation $\mathbb{E}$ is the integration in (u_{ik}) and (v_{lj}) on $U(n) \times U(n)$, where $U(n)$ is the $n \times n$ unitary group equipped with Haar measure.

Proof. The proof is similar to that of Theorem 7.1. Instead of the noncommutative Burkholder inequality via Theorem 6.5, we now use the noncommutative Khintchine inequality with help of the classical fact that (u_{ik}) can be replaced by a Gaussian matrix $n^{-1/2}\,(g_{ij})$, where the g_{ij} are independent Gaussian variables of mean-zero and variance 1 (see [MP]). Thus

$$\Big(\mathbb{E}\Big\| \sum_{i,j,k,l=1}^{n} u_{ik}v_{lj}\, a_{ij} \otimes x_{kl} \Big\|_p\Big)^{1/p} \sim_{c} \frac{1}{n}\Big(\mathbb{E}\Big\| \sum_{i,j,k,l=1}^{n} g_{ik}\, g'_{lj}\, a_{ij} \otimes x_{kl} \Big\|_p\Big)^{1/p}.$$

It then remains to repeat the arguments in the proof of Theorem 7.1 by using

$$\Big(\mathbb{E}\Big\| \sum_{i,k} g_{ik}\, y_{ik} \Big\|_p\Big)^{1/p} \sim_{c\sqrt{p}} \Big\| \Big(\sum_{i,k} y_{ik}^* y_{ik}\Big)^{1/2} \Big\|_p + \Big\| \Big(\sum_{i,k} y_{ik} y_{ik}^*\Big)^{1/2} \Big\|_p$$

for any y_{ik} in a noncommutative L_p (see [P1]). ∎

We say that (x_{ij}) is a bisymmetric basis of a subspace $X \subset L_p(\mathcal{M})$ if every entry x_{ij} is nonzero, the linear span of the x_{ij} is dense in X and there exists a constant λ such that

$$\Big\| \sum_{i,j} \varepsilon_i \varepsilon'_j a_{\pi(i)\,\pi'(j)} x_{ij} \Big\|_p \le \lambda \Big\| \sum_{i,j} a_{ij} x_{ij} \Big\|_p$$

holds for all finite scalar matrices (a_{ij}), all $\varepsilon_i = \pm 1$, $\varepsilon'_j = \pm 1$ and all permutations π and π'. It is easy to check that (x_{ij}) is indeed a basis of X according to an appropriate order, for instance, the one defined as follows. Let $e_1 = x_{11}$ and assume defined $e_1, \ldots, e_{n^2}$. Then we set $e_{n^2+j} = x_{n+1,j}$ for $j = 1, \ldots, n+1$, and $e_{n^2+n+1+i} = x_{n+1-i,n+1}$ for $i = 1, \ldots, n$. Similarly, we define completely bisymmetric bases by replacing scalar coefficients by matrices.

Recall that $\ell_p(\ell_2)$ denotes the space of all scalar matrices $a = (a_{ij})$ such that

$$\Big(\sum_i \Big(\sum_j |a_{ij}|^2 \Big)^{p/2} \Big)^{1/p} < \infty$$

and is equipped with the natural norm. $\ell_p(\ell_2)^t$ is the space of all a such that $a^t \in \ell_p(\ell_2)$, where a^t denotes the transpose of a. In the operator space setting, these spaces yield four different spaces $\ell_p(C_p), \ell_p(R_p), \ell_p(C_p)^t$ and $\ell_p(R_p)^t$, corresponding respectively to the norms γ_k for $1 \le k \le 4$ introduced at the beginning of the present section. Accordingly, the last four norms there give four other operator spaces $C_p(\mathbb{N}^2), R_p(\mathbb{N}^2), S_p$ and S_p^t. The following is the matrix analogue of Corollary 6.8. The proof is almost identical to that of Corollary 6.8 but now via Theorem 7.1.

Corollary 7.3: *Let $2 \le p < \infty$ and $X \subset L_p(\mathcal{M})$ be a subspace.*

(i) *If X has a bisymmetric basis given by an infinite matrix, then X is isomorphic to one of the following spaces*

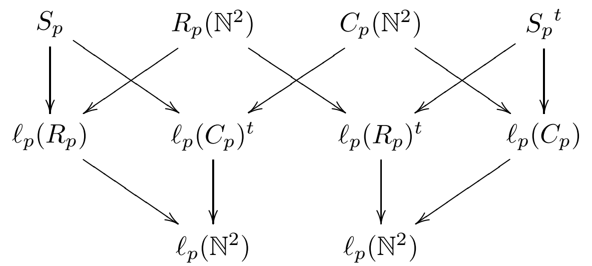
$$\ell_p(\mathbb{N}^2),\ \ell_p(\ell_2),\ \ell_p(\ell_2)^t,\ \ell_p(\ell_2) \cap \ell_p(\ell_2)^t,\ S_p,\ \ell_2(\mathbb{N}^2)\,.$$

(ii) *If X has a completely bisymmetric basis given by an infinite matrix, then X is completely isomorphic to one of the following spaces*

$$\ell_p(\mathbb{N}^2),\ \ell_p(C_p),\ \ell_p(R_p),\ \ell_p(C_p)^t,\ \ell_p(R_p)^t,\ S_p,\ {S_p}^t,\ C_p(\mathbb{N}^2),\ R_p(\mathbb{N}^2)$$

or one possible intersection of them.

Remark 7.4: The relations between the 9 building blocks in (ii) above are shown by the following diagram



The arrows indicate complete contractions, e.g., $S_p \subset \ell_p(R_p) \cap \ell_p(C_p)^t$ and $C_p(\mathbb{N}^2) \subset \ell_p(C_p) \cap \ell_p(C_p)^t$. Thus not all intersections of these spaces are nontrivial for some of them simplify. However, the four spaces on each of the first two levels do not give any nontrivial intersection, so yield 16 pairwise distinct spaces.

Remark 7.5: It is easy to see that all spaces appearing in the preceding corollary (completely) embed really into a noncommutative L_p. Note that an interesting embedding of $\ell_p(\ell_2) \cap \ell_p(\ell_2)^t$ (or $\ell_p(R_p) \cap \ell_p(C_p)^t$ in the operator space case) is given by the noncommutative Khintchine inequality in Remark 3.4.

A bisymmetric basis (x_{ij}) of X is called (completely) unitary invariant if

$$\Big\| \sum_{ij} u_{ij} a_{ij} x_{ij} \Big\|_p \le \lambda \Big\| \sum_{ij} a_{ij} x_{ij} \Big\|_p$$

holds for (a_{ij}) in $\mathbb{C}$ (in S_p). Recall that if E is a separable symmetric sequence space, the associated unitary ideal S_E is defined to be the closure of finite matrices with respect to the norm

$$\|a\|_{S_E} = \|(s_k(a))_k\|_E \, ,$$

where $(s_k(a))_k$ is the sequence of the singular numbers of a. It is well-known that the matrix units of $B(\ell_2)$ form a unitary invariant basis of S_E.

Corollary 7.6: *Let $2 \le p < \infty$ and $X \subset L_p(\mathcal{M})$ be a subspace.*

(i) *If X has a unitary invariant basis, then X is isomorphic to S_p or S_2. Consequently, a unitary ideal S_E embeds in $L_p(\mathcal{M})$ if and only if $E = \ell_p$ or $E = \ell_2$.*

(ii) *If X has a completely unitary invariant basis, then X is completely isomorphic to one of the* 16 *spaces:* S_p, ${S_p}^t$, $C_p(\mathbb{N}^2)$, $R_p(\mathbb{N}^2)$ *and their intersections.*

Proof. This is an immediate consequence of the preceding corollary since all spaces there but those in the present corollary are not unitary invariant. Alternately, we can also follow the proof of Corollary 7.3 by using Theorem 7.2. ∎

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